Advanced quantum information: entanglement and nonlocality

2. Homework sheet

Solutions to be submitted via email to m.scalici@cent.uw.edu.pl
Please submit a single pdf file using "Solutions Advanced Quantum Information" in the subject line. Latest date for submission: 5. April 2022

Problem 1 (Lecture notes Section 4)

Assume that Alice and Bob share a Bell state

$$|\Phi^+\rangle^{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Additionally, Alice has two qubits C and D in a general state

$$|\psi\rangle^{CD} = \sum_{i,j} c_{ij} |i\rangle^C \otimes |j\rangle^D , \tag{1}$$

where c_{ij} are complex coefficients such that $\sum_{i,j} |c_{ij}|^2 = 1$. Convince yourself by explicit calculation that Alice can teleport the qubit D to Bob by using the following procedure:

1. Alice performs a measurement on her qubits A and D in the maximally entangled basis $\{|\Phi^+\rangle^{AD}, |\Phi^-\rangle^{AD}, |\Psi^+\rangle^{AD}, |\Psi^-\rangle^{AD}\}$. In her measurement, Alice obtains one of four possible outcomes, corresponding to one of the Bell states. Give an explicit formula for each of the post-measurement states $|\mu\rangle^{ABCD}$ of all the qubits.

2. Bob performs a conditional unitary on his qubit depending on Alice's measurement outcome, as given in the following table (σ_i are Pauli matrices):

Alice's outcome	$\lvert\Phi^+\rangle$	$\lvert\Phi^-\rangle$	$\lvert\Psi^+\rangle$	$\lvert\Psi^-\rangle$
Bob's unitary	$\mathbb{1}$	σ_z	σ_x	$-i\sigma_y$

   Give an explicit expression for the final state $|\nu\rangle^{ABCD}$ of all the qubits for each case.

Problem 2 (Lecture notes Section 5)

a) Assume that Alice and Bob share a quantum state $|\psi\rangle^{AB}$ which has the Schmidt decomposition

$$|\psi\rangle^{AB} = \sum_{i=0}^{s-1} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle ,$$

where s is the number of non-zero Schmidt components, also called the <u>Schmidt number</u>. Let now Alice and Bob apply an LOCC protocol transforming $|\psi\rangle^{AB}$ into another pure state $|\phi\rangle^{AB}$. Prove that the Schmidt number cannot increase in this process.

b) Assume now that Alice and Bob share m copies of the Bell state $|\Phi^+\rangle$. Use the arguments from part a) to show that there is no LOCC protocol such that

$$|\Phi^+\rangle^{\otimes m} \rightarrow |\Phi^+\rangle^{\otimes n}$$

with $n > m$.

c) In Propositions 5.1 and 5.2 of the lecture notes we have proven that the entanglement cost of a state $|\psi\rangle$ is at most $S(\rho_\psi)$ and that the distillable entanglement of $|\psi\rangle$ is at least $S(\rho_\psi)$. We then used these results in Theorem 5.1 to prove that the distillable entanglement must be equal to $S(\rho_\psi)$. Prove that the entanglement cost is equal to $S(\rho_\psi)$.

Hint: the arguments are similar to the proof of Theorem 5.1.

d) Use the arguments from part c) to prove that the optimal rate for converting a state $|\psi\rangle$ into an entangled state $|\phi\rangle$ via LOCC is given by

$$r = \frac{S(\rho_\psi)}{S(\rho_\phi)},$$

i.e., for m and n large enough there exists an LOCC protocol such that

$$|\psi\rangle^{\otimes m} \rightarrow |\phi\rangle^{\otimes n}$$

and $n/m \approx r$.

Hint: consider first conversion $|\psi\rangle \rightarrow |\Phi^+\rangle$, and then the conversion $|\Phi^+\rangle \rightarrow |\phi\rangle$.

e) Assume that Alice and Bob share a state $|\psi\rangle^{AB}$. Show that whenever $|\psi\rangle^{AB}$ is entangled Alice and Bob can obtain a Bell state $|\Phi^+\rangle$ with nonzero probability by using LOCC. This proves that all pure entangled states are single-copy distillable.

f) For $d_A = d_B = 3$ consider the following state for $0 \leq a \leq 1$:

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}.$$

Prove that ρ_a has positive partial transpose for $0 \leq a \leq 1$. Show numerically that the realigned matrix $\widetilde{\rho_a}$ fulfills $\|\widetilde{\rho_a}\|_1 > 1$ for all $0 < a < 1$. This proves that the state ρ_a is bound entangled in the range $0 < a < 1$.