Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

1st class
March 2, 2022

Advanced quantum information

- Every Wednesday 15:15 - 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al.*, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- Homework to be submitted via email as a single pdf

Outline

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Quantum states

- Any physical system is completely described by a **state vector** $|\psi\rangle \in \mathcal{H}$

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- **Qubit:** quantum system with Hilbert space dimension 2
- A system which is described by a single state vector is in a **pure state**

Quantum states

- A system which is in the pure state $|\psi_i\rangle$ with probability p_i is described by a **density matrix**

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

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- If $p_{\max} < 1$, the system is in a **mixed state**

Quantum states

Example: Consider $p_0 = p_1 = 1/2$ and

$$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$|\psi_1\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

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The density matrix is given by

$$\begin{aligned}
\rho &= \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|\psi_1\rangle\langle\psi_1| \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+\cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}
\end{aligned}$$

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Second property also implies that ρ is Hermitian: $\rho^\dagger = \rho$.

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- The post-measurement state of the particle is either $|\uparrow\rangle$ or $|\downarrow\rangle$

Quantum measurements and operations

- **General quantum measurement:** collection $\{K_i\}$ of Kraus operators that fulfill the completeness equation:

$$\sum_i K_i^\dagger K_i = \mathbb{1}_d = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \tag{1}$$

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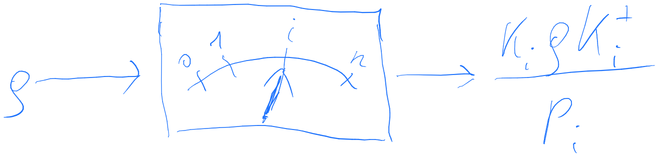
- Post-measurement state of the system after occurrence of outcome i:

$$\rho_i = \frac{K_i \rho K_i^\dagger}{p_i}$$

Quantum measurements and operations



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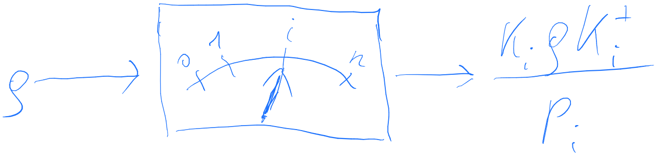

- Any set of Kraus operators corresponds to a measurement, in principle realizable in laboratory
- For any physically realizable measurement there exists a valid set of Kraus operators

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- Probabilities of the outcomes:

$$p_i = \mathrm{Tr}[M_i \rho]$$

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- **Von Neumann measurement:** K_i are orthogonal projectors with rank one

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- They correspond to a special class of linear maps, which are completely positive and trace preserving (CPTP)

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Composite systems

- Consider two parties, Alice and Bob, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$

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- Consider two parties, Alice and Bob, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Total Hilbert space is a **tensor product** of the subsystem spaces:
$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Composite systems

Example:

- Consider the states

$$|\psi\rangle^A = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

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- The state of the total system is

$$|\psi\rangle^{AB} = |\psi\rangle^{A} \otimes |\psi\rangle^{B} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \otimes \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta \\ \cos\alpha\sin\beta \\ \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \end{pmatrix}$$

Composite systems

- If $\{|i\rangle\}$ and $\{|k\rangle\}$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, then $\{|i\rangle \otimes |k\rangle\}$ is an **orthonormal basis** of $\mathcal{H}_{AB}$

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$$|\psi\rangle^{AB} = \sum_{i,k} c_{ik} \, |i\rangle \otimes |k\rangle \, .$$

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- Any **density matrix** can be expanded as

$$\rho^{AB} = \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |k\rangle\langle l|$$

  with $c_{ijkl} \in \mathbb{C}$

Composite systems

- Subsystem A is described by the **reduced density matrix**

$$\rho^A = \mathrm{Tr}_B[\rho^{AB}] = \sum_{i,j,k,l} c_{ijkl} |i\rangle\langle j| \, \mathrm{Tr}\left[|k\rangle\langle l|\right]$$

$$= \sum_{i,j,k,l} c_{ijkl} |i\rangle\langle j| \, \delta_{kl} = \sum_{i,j,k} c_{ijkk} |i\rangle\langle j|$$

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- Tr_B is the **partial trace** over the subsystem B

Composite systems

Example: Consider the density matrix

$$\rho^{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

with matrices $X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

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The reduced density matrices are

$$\rho^{A} = \begin{pmatrix} \mathrm{Tr}\left[X\right] & \mathrm{Tr}\left[Y\right] \\ \mathrm{Tr}\left[Y^\dagger\right] & \mathrm{Tr}\left[Z\right] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho^{B} = X + Z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Composite systems

- **Schmidt decomposition:** For any pure state $|\psi\rangle^{AB}$ there exists a product basis $\{|i\rangle \otimes |j\rangle\}$ such that

$$|\psi\rangle^{AB} = \sum_i \sqrt{\lambda_i}\, |i\rangle \otimes |i\rangle$$

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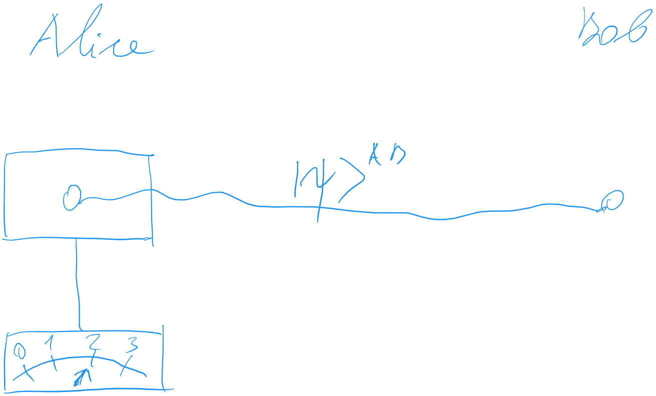
- The numbers λ_i are called **Schmidt coefficients** of $|\psi\rangle^{AB}$

Composite systems



- Kraus operators of local measurements: $K_i^{AB} = K_i \otimes \mathbb{1}$

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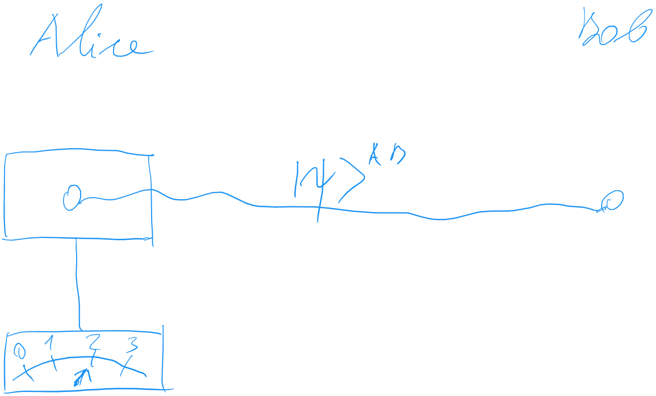

- Kraus operators of local measurements: $K_i^{AB} = K_i \otimes \mathbb{1}$
- Completeness condition: $\sum_i \left(K_i^{AB}\right)^\dagger K_i^{AB} = \sum_i K_i^\dagger K_i \otimes \mathbb{1} = \mathbb{1}_{AB}$

Composite systems

Local quantum operations on Alice's side:

$$\Lambda^A(\rho^{AB}) = \sum_i (K_i \otimes \mathbb{1})\, \rho^{AB}\, (K_i \otimes \mathbb{1})^\dagger$$

Composite systems

State of Bob does not change upon local operations of Alice:

$$\rho^B = \mathrm{Tr}_A \left[\rho^{AB}\right] = \mathrm{Tr}_A \left[\Lambda^A(\rho^{AB})\right]$$

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- Two states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ are purifications of the same state ρ^A if and only if

$$|\psi\rangle^{AB} = (\mathbb{1} \otimes U)\,|\phi\rangle^{AB}$$

  for some local unitary U

Useful properties of square matrices

- **Functions of matrices:** Let f be a function from $\mathbb{C}$ to $\mathbb{C}$. For a normal (diogonalizable) matrix $A = \sum_i a_i |\psi_i\rangle\langle\psi_i|$ with eigenvalues $a_i \in \mathbb{C}$ and eigenstates $|\psi_i\rangle$ we define

$$f(A) := \sum_i f(a_i) |\psi_i\rangle\langle\psi_i|$$

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- **Polar decomposition:** For any square matrix A there exist unitary matrices U and V such that

$$A = U\sqrt{A^\dagger A} = \sqrt{AA^\dagger}V$$

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Theory of quantum entanglement

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- $|\psi\rangle^{AB}$ is product if and only if ρ^A is pure

Theory of quantum entanglement

Exercise: Consider the state

$$|\psi\rangle^{AB} = \frac{1}{\sqrt{7}}\left(2\,|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

Is the state entangled or separable?

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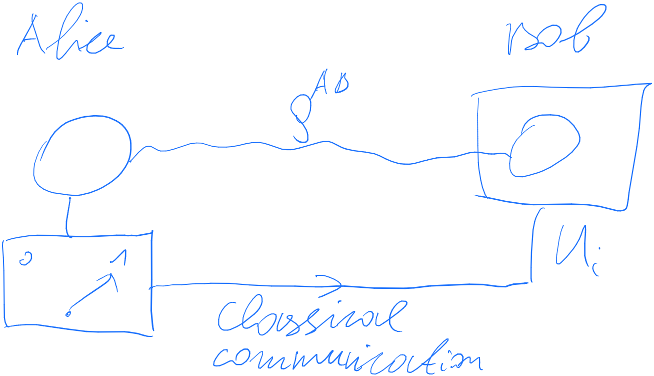
Solution: Density matrix $\rho^{AB} = |\psi\rangle\langle\psi|^{AB}$ is given by

$$\rho^{AB} = \frac{1}{7}\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \rho^{A} = \frac{1}{7}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

Determinant of ρ^A is given by $1/49 \Rightarrow |\psi\rangle^{AB}$ is entangled

Outline

Local operations and classical communication (LOCC)



LOCC describes the most general procedure Alice and Bob can apply, if they can perform arbitrary quantum measurements/operations locally, and exchange classical information

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4. The outcome j of Bob's measurement is communicated classically to Alice.
5. Alice performs a local measurement on her subsystem which can depend on all outcomes of all previous measurements, and the process starts over at step 2.