Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

6th class
April 6, 2022

Advanced quantum information (6th class)

- Every Wednesday 15:15 – 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al*., *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes (will be updated today): http://qot.cent.uw.edu.pl/teaching/
- 3. Homework sheet to be submitted via email by 19. April

Outline

1. Homework problems

2. Entangelment measures
   - Trace distance and fidelity
   - Distance-based entanglement measures

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1. Homework problems

2. Entangement measures
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   - Distance-based entanglement measures

Sheet 2, Problem 2 a)

Problem 2 a) Assume that Alice and Bob share a quantum state $|\psi\rangle^{AB}$ which has the Schmidt decomposition

$$|\psi\rangle^{AB} = \sum_{i=0}^{s-1} \sqrt{\lambda_i}\, |i\rangle \otimes |i\rangle\,,$$

where s is the number of non-zero Schmidt components, also called the Schmidt number. Let now Alice and Bob apply an LOCC protocol transforming $|\psi\rangle^{AB}$ into another pure state $|\phi\rangle^{AB}$. Prove that the Schmidt number cannot increase in this process.

Sheet 2, Problem 2 a)

Solution: Consider

$$|\psi\rangle^{AB} = \sum_{i=0}^{s-1} \sqrt{\lambda_i}\,|i\rangle \otimes |i\rangle\,, \qquad |\phi\rangle^{AB} = \sum_{i=0}^{s'-1} \sqrt{\mu_i}\,|i\rangle \otimes |i\rangle$$

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- Assume (by contradiction) that there exists an LOCC protocol converting $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ with $s' > s$

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- Define vectors (sorted in decreasing order)

$$\vec{\lambda} = (\underbrace{\lambda_0, \ldots, \lambda_{s-1}, 0, \ldots 0}_{s'}),$$

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- **Contradiction**:

$$\sum_{i=0}^{s-1} \lambda_i = 1 > \sum_{i=0}^{s-1} \mu_i$$

Sheet 2, Problem 2 e)

Problem 2 e)
Assume that Alice and Bob share a state $|\psi\rangle^{AB}$. Show that whenever $|\psi\rangle^{AB}$ is entangled Alice and Bob can obtain a Bell state $|\Phi^+\rangle$ with nonzero probability by using LOCC. This proves that all pure entangled states are single-copy distillable.

Sheet 2, Problem 2 e)

Solution: Consider Schmidt decomposition

$$|\psi\rangle^{AB} = \sqrt{\lambda_0}\,|\alpha_0\rangle\,|\beta_0\rangle + \sqrt{1-\lambda_0}\,|\alpha_1\rangle\,|\beta_1\rangle$$

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Alice applies local measurement with Kraus operators

$$K_0 = \sqrt{\frac{1-\lambda_0}{\lambda_0}}\,|0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|, \qquad K_1 = \sqrt{\frac{2\lambda_0 - 1}{\lambda_0}}\,|0\rangle\langle\alpha_0|$$

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Note that

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \frac{1-\lambda_0}{\lambda_0}\,|\alpha_0\rangle\langle\alpha_0| + |\alpha_1\rangle\langle\alpha_1| + \frac{2\lambda_0 - 1}{\lambda_0}\,|\alpha_0\rangle\langle\alpha_0| = \mathbb{1}$$

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- Probability of outcome 0: $p_0 = \langle\psi|K_0^\dagger K_0 \otimes \mathbb{1}|\psi\rangle = 2(1-\lambda_0) > 0$

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- Post-measurement state:
  $\frac{1}{\sqrt{p_0}} K_0 \otimes \mathbb{1}\,|\psi\rangle^{AB} = \frac{1}{\sqrt{2}}\,(|0\rangle\,|\beta_0\rangle + |1\rangle\,|\beta_1\rangle)$

Sheet 2, Problem 2 e)

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Bob applies local unitary $U = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$
$\Rightarrow$ Alice and Bob end with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Sheet 2, Problem 2 f)

Problem 2 f) For $d_A = d_B = 3$ consider the following state for $0 \leq a \leq 1$:

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}.$$

Prove that ρ_a has positive partial transpose for $0 \leq a \leq 1$. Show numerically that the realigned matrix $\widetilde{\rho_a}$ fulfills $\|\widetilde{\rho_a}\|_1 > 1$ for all $0 < a < 1$. This proves that the state ρ_a is bound entangled in the range $0 < a < 1$.

Sheet 2, Problem 2 f)

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$$\rho_a^{T_B} = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

Sheet 2, Problem 2 f)

Eigenvalues of $\rho_a^{T_B}$:

$$\frac{1+2a-\sqrt{1-2a+2a^2}}{2(1+8a)},$$
$$\frac{1+2a+\sqrt{1-2a+2a^2}}{2(1+8a)},$$
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Note that

$$(1+2a)^2-\left[1-2a+2a^2\right]=2a(3+a)\geq 0$$

$\Rightarrow \rho_a^{T_B}$ has no negative eigenvalues

Sheet 2, Problem 2 f)

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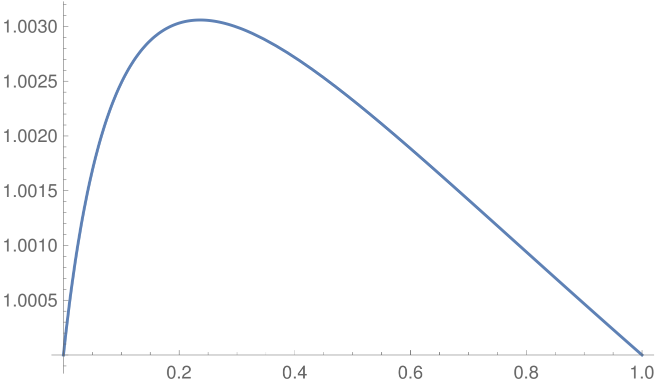

$\|\widetilde{\rho_a}\|_1$ as function of a

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- $\Rightarrow \rho_a$ is bound entangled

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How much entanglement is in a given quantum state ρ?

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2. E does not increase under local operations and classical communication:

$$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho)$$

   for any LOCC protocol Λ_{LOCC}

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Many entanglement measures have additional features:

- Convexity $E\left(\sum_i p_i \rho_i^{AB}\right) \leq \sum_i p_i E\left(\rho_i^{AB}\right)$
- Strong monotonicity: $\sum_i q_i E\left(\sigma_i^{AB}\right) \leq E\left(\rho^{AB}\right)$

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Minimum is taken over all decompositions $\{p_i, |\psi_i\rangle^{AB}\}$ such that $\rho^{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|^{AB}$

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- E_f is convex and fulfills strong monotonicity
- Exact expression can be given for 2-qubit states

Outline

Trace distance

- **Trace distance**:

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- Decompositition of H into orthogonal parts: $H = P_+ - P_-$ with positive matrices $P_\pm$

Trace distance

Theorem 6.2. For any Hermitian $d \times d$ matrix H and any trace preserving positive linear map Λ acting on the Hilbert space of dimension d it holds that

$$\left\| \Lambda \left(H \right) \right\|_1 \leq \left\| H \right\|_1 \tag{1}$$

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Theorem 6.2. For any Hermitian $d \times d$ matrix H and any trace preserving positive linear map Λ acting on the Hilbert space of dimension d it holds that

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Proof.

- Let $\Lambda(H) = Q_+ - Q_-$ and $H = P_+ - P_-$ be decompositions into orthogonal parts $Q_\pm \geq 0$ and $P_\pm \geq 0$

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- **Exercise:** prove that

$$\mathrm{Tr}\left(Q_+\right) \leq \mathrm{Tr}\left(\Lambda\left[P_+\right]\right)$$
$$\mathrm{Tr}\left(Q_-\right) \leq \mathrm{Tr}\left(\Lambda\left[P_-\right]\right)$$

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- **Solution:** Let Π_+ be the projector onto the subslace of Q_+. From $Q_+ - Q_- = \Lambda\left(P_+\right) - \Lambda\left(P_-\right)$ we obtain

$$\begin{aligned} \mathrm{Tr}\left(Q_+\right) &= \mathrm{Tr}\left(\Pi_+\left[Q_+ - Q_-\right]\right) \\ &= \mathrm{Tr}\left(\Pi_+\Lambda\left[P_+\right]\right) - \mathrm{Tr}\left(\Pi_+\Lambda\left[P_-\right]\right) \leq \mathrm{Tr}\left(\Lambda\left[P_+\right]\right) \end{aligned}$$

and similarly for $\mathrm{Tr}\left(Q_-\right)$

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Proof.

$$\Lambda\left(H\right) = Q_+ - Q_-, \quad H = P_+ - P_-,$$
$$\mathrm{Tr}\left(Q_\pm\right) \leq \mathrm{Tr}\left(\Lambda\left[P_\pm\right]\right)$$

Recalling that Λ is trace preserving, we further have

$$\mathrm{Tr}\left(Q_+ + Q_-\right) \leq \mathrm{Tr}\left(P_+ + P_-\right)$$

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The proof is complete using the fact that

$$\|H\|_1 = \mathrm{Tr}(P_+ + P_-), \quad \|\Lambda(H)\|_1 = \mathrm{Tr}(Q_+ + Q_-)$$

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Proposition 6.5. For any unitary U it holds that

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Proof. By the polar decomposition we have

$$\left|\mathrm{Tr}\left(AU\right)\right| = \left|\mathrm{Tr}\left(V\sqrt{A^\dagger A}\,U\right)\right| = \left|\mathrm{Tr}\left(\left[A^\dagger A\right]^{1/4}\left[A^\dagger A\right]^{1/4} UV\right)\right|.$$

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$$\left|\mathrm{Tr}\,(AU)\right| \leq \|A\|_1$$

Proof. By the polar decomposition we have

$$\left|\mathrm{Tr}\,(AU)\right| = \left|\mathrm{Tr}\left(V\sqrt{A^\dagger A}\,U\right)\right| = \left|\mathrm{Tr}\left(\left[A^\dagger A\right]^{1/4}\left[A^\dagger A\right]^{1/4}UV\right)\right|.$$

Using the Cauchy-Schwarz inequality $\left|\mathrm{Tr}\left(X^\dagger Y\right)\right|^2 \leq \mathrm{Tr}\left(X^\dagger X\right)\mathrm{Tr}\left(Y^\dagger Y\right)$ and setting

$$X = \left[A^\dagger A\right]^{1/4}, \quad Y = \left[A^\dagger A\right]^{1/4} UV$$

we obtain

$$\left|\mathrm{Tr}\,(AU)\right| \leq \sqrt{\mathrm{Tr}\,\sqrt{A^\dagger A}\,\mathrm{Tr}\left(V^\dagger U^\dagger \sqrt{A^\dagger A}\,UV\right)} = \mathrm{Tr}\,\sqrt{A^\dagger A} = \|A\|_1\,.$$

Q.E.D.

Fidelity

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$$F(\rho, \sigma) = \mathrm{Tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$$

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- It follows that $0 \leq F(\rho, \sigma) \leq 1$, and $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$
- Let $|\psi\rangle = |\psi\rangle^{AB}$ and $|\phi\rangle = |\phi\rangle^{AB}$ be purifications $\rho = \rho^B$ and $\sigma = \sigma^B$

Fidelity

Theorem 6.3. For any two states ρ and σ it holds that

$$F(\rho, \sigma) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle| ,$$

where the maximum is taken over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ.

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Proof. We can write any purification of ρ and σ as follows:

$$|\psi\rangle = (U_A \otimes \sqrt{\rho} U_B) |m\rangle,$$
$$|\phi\rangle = \left(V_A \otimes \sqrt{\sigma} V_B\right) |m\rangle$$

with $d_A = d_B$, $|m\rangle = \sum_i |i\rangle |i\rangle$ and some unitaries U_A, U_B, V_A, and V_B.

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with $d_A = d_B$, $|m\rangle = \sum_i |i\rangle |i\rangle$ and some unitaries U_A, U_B, V_A, and V_B.
We obtain

$$|\langle\psi|\phi\rangle| = \left|\langle m|U_A^\dagger V_A \otimes U_B^\dagger \sqrt{\rho}\sqrt{\sigma} V_B|m\rangle\right| .$$

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Proof.

$$|\langle\psi|\phi\rangle| = \left|\langle m| U_A^\dagger V_A \otimes U_B^\dagger \sqrt{\rho}\sqrt{\sigma} V_B |m\rangle\right|$$

Using the equality

$$|\langle m| A \otimes B |m\rangle| = \mathrm{Tr}\left(A^\dagger B\right)$$

we further obtain

$$|\langle\psi|\phi\rangle| = \left|\mathrm{Tr}\left(V_A^\dagger U_A U_B^\dagger \sqrt{\rho}\sqrt{\sigma} V_B\right)\right|.$$

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Proof.

$$|\langle\psi|\phi\rangle| = \left|\mathrm{Tr}\left(V_A^\dagger U_A U_B^\dagger \sqrt{\rho}\sqrt{\sigma} V_B\right)\right|$$

Defining the unitary $U = V_B V_A^\dagger U_A U_B^\dagger$ we arrive at

$$|\langle\psi|\phi\rangle| = \left|\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\sigma} U\right)\right|.$$

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Defining the unitary $U = V_B V_A^\dagger U_A U_B^\dagger$ we arrive at

$$|\langle\psi|\phi\rangle| = \left|\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\sigma} U\right)\right|.$$

Using Proposition 6.5., we see that

$$|\langle\psi|\phi\rangle| \leq \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = F(\rho, \sigma).$$

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where the maximum is taken over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ.

Proof.

$$|\langle\psi|\phi\rangle| = \left|\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\sigma} V_B V_A^\dagger U_A U_B^\dagger\right)\right| \leq \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = F(\rho, \sigma)$$

Equality can be attained by choosing a unitary V such that $M = \sqrt{MM^\dagger} V$, where $M = \sqrt{\rho}\sqrt{\sigma}$. Setting $V_B = V^\dagger$ and $U_B = U_A = V_A = \mathbb{1}$ the equality is attained.
Q.E.D.

Fidelity

Theorem 6.4. For any two quantum states ρ and σ and any quantum operation Λ it holds that

$$F\left(\Lambda\left[\rho\right], \Lambda\left[\sigma\right]\right) \geq F\left(\rho, \sigma\right)$$

Fidelity

Theorem 6.4. For any two quantum states ρ and σ and any quantum operation Λ it holds that

$$F(\Lambda[\rho], \Lambda[\sigma]) \geq F(\rho, \sigma)$$

Proof.

- Every quantum operation Λ on the system B can be written as

$$\Lambda\left[\rho^B\right] = \mathrm{Tr}_E\left[U_{BE}\left(\rho^B \otimes |0\rangle\langle 0|^E\right) U_{BE}^\dagger\right]$$

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- Let $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ be purifications of ρ^B and σ^B, such that $F(\rho, \sigma) = |\langle\psi|\phi\rangle|$

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- Let $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ be purifications of ρ^B and σ^B, such that $F(\rho,\sigma) = |\langle\psi|\phi\rangle|$
- $\mathbb{1} \otimes U_{BE}\,|\psi\rangle^{AB}\,|0\rangle^E$ is a purification of $\Lambda[\rho^B]$ and $\mathbb{1} \otimes U_{BE}\,|\phi\rangle^{AB}\,|0\rangle^E$ is a purification of $\Lambda[\sigma^B]$

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- $\mathbb{1} \otimes U_{BE} \left|\psi\right\rangle^{AB} \left|0\right\rangle^{E}$ is a purification of $\Lambda[\rho^B]$ and $\mathbb{1} \otimes U_{BE} \left|\phi\right\rangle^{AB} \left|0\right\rangle^{E}$ is a purification of $\Lambda[\sigma^B]$
- Using Theorem 6.3 we obtain

$$F\left(\Lambda\left[\rho\right], \Lambda\left[\sigma\right]\right) \geq \left|\left\langle\psi\right| \left\langle 0\right| \mathbb{1} \otimes U_{BE}^{\dagger} U_{BE} \left|\phi\right\rangle \left|0\right\rangle\right| = \left|\left\langle\psi|\phi\right\rangle\right| = F(\rho, \sigma)$$

Q.E.D.

Fidelity

- **Bure distance:**

$$D_b(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}$$

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Fidelity

- **Bure distance:**

$$D_b(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}$$

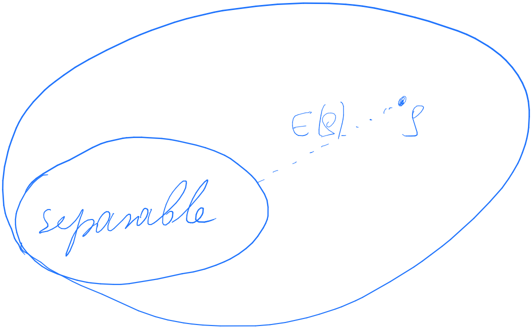
- $D_b(\rho, \sigma) \geq 0$ with equality if and only if $\rho = \sigma$
- D_b fulfills the data-processing inequality

$$D_b(\Lambda[\rho], \Lambda[\sigma]) \leq D_b(\rho, \sigma)$$

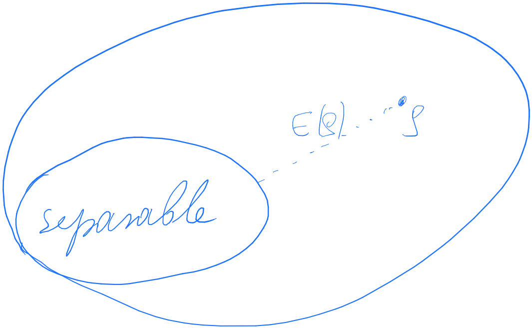
  for any quantum operation Λ

Outline

Distance-based entanglement measures



Distance-based entanglement measures

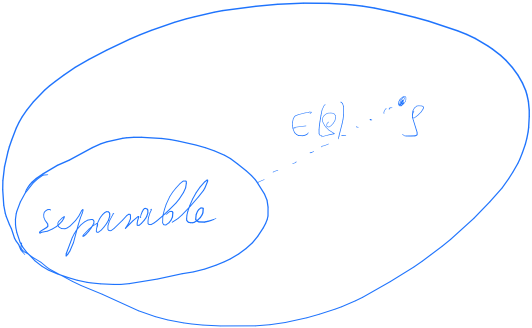


For a distance function $D(\rho, \sigma)$ define

$$E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$$

with infimum over separable states $\mathcal{S}$

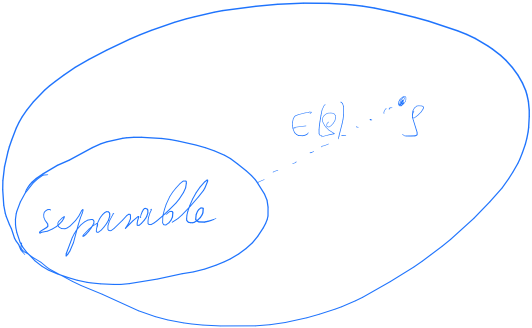
Distance-based entanglement measures



E is an entanglement measure if:

1. $D(\rho, \sigma) \geq 0$ with equality for $\rho = \sigma$

Distance-based entanglement measures



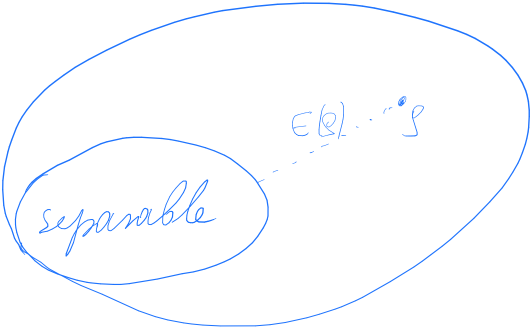
E is an entanglement measure if:

1. $D(\rho, \sigma) \geq 0$ with equality for $\rho = \sigma$
2. D fulfills the data-processing inequality:

$$D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$$

for any quantum operation Λ

Distance-based entanglement measures



Exercise: prove that

$$E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$$

is an entanglement measure

Distance-based entanglement measures

Proof that $E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$ does not increase under LOCC:

$$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho)$$

Distance-based entanglement measures

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- Let σ be a separable state such that $E(\rho) = D(\rho, \sigma)$

Distance-based entanglement measures

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- Note that $\Lambda_{\mathrm{LOCC}}[\sigma]$ is separable

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$$E(\Lambda_{\text{LOCC}}[\rho]) \leq E(\rho)$$

- Let σ be a separable state such that $E(\rho) = D(\rho, \sigma)$
- Note that $\Lambda_{\text{LOCC}}[\sigma]$ is separable
- We have

$$\begin{aligned} E\left(\Lambda_{\text{LOCC}}[\rho]\right) &= \min_{\mu \in \mathcal{S}} D\left(\Lambda_{\text{LOCC}}[\rho], \mu\right) \leq D\left(\Lambda_{\text{LOCC}}[\rho], \Lambda_{\text{LOCC}}[\sigma]\right) \\ &\leq D(\rho, \sigma) = E(\rho) \end{aligned}$$

Distance-based entanglement measures

Proof that $E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$ does not increase under LOCC:

$$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho)$$

- Let σ be a separable state such that $E(\rho) = D(\rho, \sigma)$
- Note that $\Lambda_{\mathrm{LOCC}}[\sigma]$ is separable
- We have

$$\begin{aligned} E\left(\Lambda_{\mathrm{LOCC}}[\rho]\right) &= \min_{\mu \in \mathcal{S}} D\left(\Lambda_{\mathrm{LOCC}}[\rho], \mu\right) \leq D\left(\Lambda_{\mathrm{LOCC}}[\rho], \Lambda_{\mathrm{LOCC}}[\sigma]\right) \\ &\leq D(\rho, \sigma) = E(\rho) \end{aligned}$$

- Proof holds also if Λ_{LOCC} is replaced by separable operations