Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

7th class
April 20, 2022

Advanced quantum information (6th class)

- Every Wednesday 15:15 – 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al.*, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Part 1 lecture materials: http://qot.cent.uw.edu.pl/teaching/
- Part 2 lecture materials: http://jkaniewski.fuw.edu.pl/?q=teaching

Outline

1. Quantification of entanglement
   - Distance-based entanglement measures
   - Negativity
   - Distillable entanglement and entanglement cost
2. Entanglement monogamy

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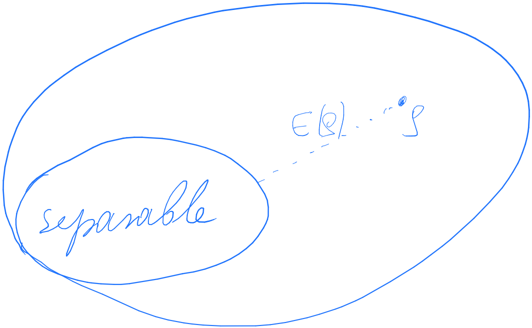
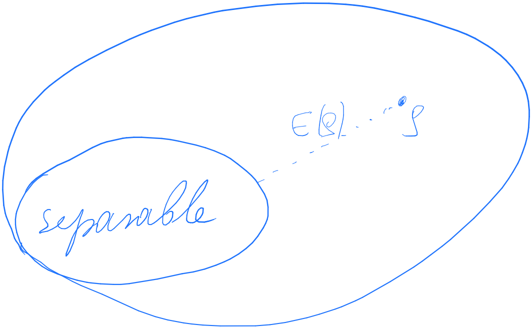
Distance-based entanglement measures



Distance-based entanglement measures

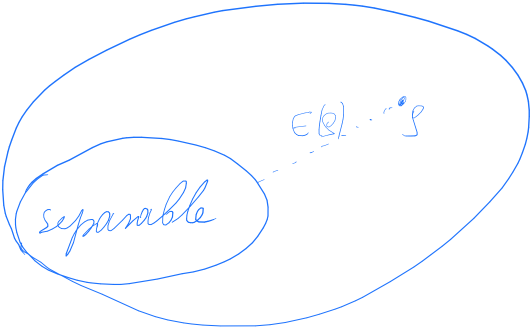


For a distance function $D(\rho, \sigma)$ define

$$E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$$

with infimum over separable states $\mathcal{S}$

Distance-based entanglement measures



E is an entanglement measure if:

1. $D(\rho, \sigma) \geq 0$ with equality for $\rho = \sigma$

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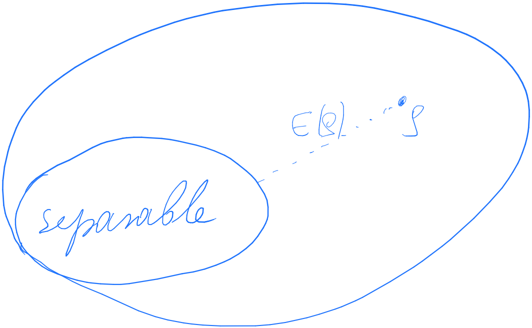


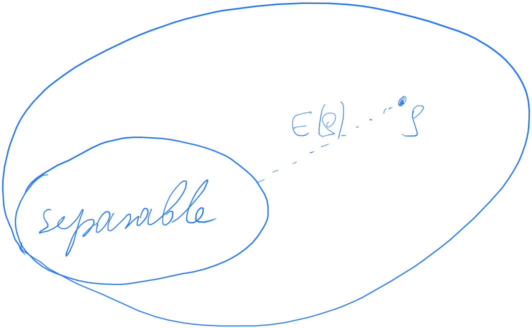
E is an entanglement measure if:

1. $D(\rho, \sigma) \geq 0$ with equality for $\rho = \sigma$
2. D fulfills the data-processing inequality:

$$D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$$

for any quantum operation Λ

Distance-based entanglement measures



Exercise: prove that

$$E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$$

is an entanglement measure

Distance-based entanglement measures

Proof that $E(\rho) = \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma)$ does not increase under LOCC:

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- We have

$$\begin{aligned} E\left(\Lambda_{\mathrm{LOCC}}[\rho]\right) &= \min_{\mu \in \mathcal{S}} D\left(\Lambda_{\mathrm{LOCC}}[\rho], \mu\right) \leq D\left(\Lambda_{\mathrm{LOCC}}[\rho], \Lambda_{\mathrm{LOCC}}[\sigma]\right) \\ &\leq D(\rho, \sigma) = E(\rho) \end{aligned}$$

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- Proof holds also if Λ_{LOCC} is replaced by separable operations

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Examples for distances fulfilling $D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$:

Distance-based entanglement measures

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- Quantum relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}[\rho \log_2 \rho] - \mathrm{Tr}[\rho \log_2 \sigma]$$

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- Relative entropy of entanglement:

$$E_r(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$$

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- Upper bound on distillable entanglement
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- For mixed states: hard to compute in general

Distance-based entanglement measures

Examples for distances fulfilling $D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$:

- Bures distance

$$D_b(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}$$

with fidelity $F(\rho, \sigma) = \text{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$

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- Trace distance

$$D_t(\rho, \sigma) = \frac{1}{2}||\rho - \sigma||_1$$

with the trace norm $||M||_1 = \mathrm{Tr}\, \sqrt{M^\dagger M}$

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Negativity

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- It holds that $E_n(\rho^{AB}) \geq 0$, and $E_n(\rho^{AB}) = 0$ if ρ^{AB} has non-negative partial transpose

Monotonicity of E_n under LOCC

Theorem 6.2. Negativity does not increase under LOCC:

$$E_n\left(\Lambda_{\mathrm{LOCC}}\left[\rho^{AB}\right]\right) \leq E_n\left(\rho^{AB}\right).$$

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- Section 5.5.: any LOCC protocol can be written as

$$\Lambda_{\mathrm{LOCC}}[\rho^{AB}] = \sum_i A_i \otimes B_i \rho^{AB} A_i^\dagger \otimes B_i^\dagger$$

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$\Rightarrow A_i \otimes B_i^*$ are also valid Kraus operators

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Proof.

- Partial transpose of $\Lambda_{\text{LOCC}}[\rho^{AB}]$:

$$\begin{aligned}\left(\Lambda_{\text{LOCC}}\left[\rho^{AB}\right]\right)^{T_B} &= \left(\sum_i A_i \otimes B_i \rho^{AB} A_i^\dagger \otimes B_i^\dagger\right)^{T_B} \\ &= \sum_i A_i \otimes B_i^* \rho^{T_B} A_i^\dagger \otimes B_i^T\end{aligned}$$

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- Taking the trace norm gives

$$\left\|\left(\Lambda_{\mathrm{LOCC}}\left[\rho^{AB}\right]\right)^{T_B}\right\|_1 = \left\|\sum_i A_i \otimes B_i^* \rho^{T_B} A_i^\dagger \otimes B_i^T\right\|_1 = \left\|\tilde{\Lambda}\left[\rho^{T_B}\right]\right\|_1$$

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- In summary:

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Recall definition of negativity:

$$E_n(\rho^{AB}) = \frac{\|\rho^{T_B}\|_1 - 1}{2}$$

Q.E.D.

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for any states σ_i^{AB} and probabilities q_i obtainable from ρ^{AB} via LOCC

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- Negativity is convex:

$$E_n\left(\sum_i p_i \rho_i^{AB}\right) \leq \sum_i p_i E_n\left(\rho_i^{AB}\right)$$

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- **Distillable entanglement:** singlet rate obtainable from a quantum state ρ via LOCC in the asymptotic limit

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- Explicit formula:

$$E_d(\rho) = \sup\left\{r : \lim_{n\to\infty}\left(\inf_{\Lambda}\left\|\Lambda\left[\rho^{\otimes n}\right] - |\Phi^+\rangle\langle\Phi^+|^{\otimes\lfloor rn\rfloor}\right\|_1\right) = 0\right\}$$

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Distillable entanglement and entanglement cost

- E_d and E_c are special cases of **asymptotic state-conversion rates**

$$R(\rho \to \sigma) = \sup\left\{r : \lim_{n\to\infty}\left(\inf_{\Lambda}\left\|\Lambda\left[\rho^{\otimes n}\right] - \sigma^{\otimes\lfloor rn\rfloor}\right\|_1\right) = 0\right\}$$

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- It holds

$$E_d(\rho) = R(\rho \to |\Phi^+\rangle\langle\Phi^+|), \qquad E_c(\rho) = \left[R(|\Phi^+\rangle\langle\Phi^+| \to \rho)\right]^{-1}$$

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- For pure states holds

$$R(|\psi\rangle \to |\phi\rangle) = \frac{S(\rho_\psi)}{S(\rho_\phi)},$$

where ρ_ψ is the reduced state of $|\psi\rangle$

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Bounds on E_d and E_c:

Distillable entanglement and entanglement cost

Bounds on E_d and E_c:

$$E_d(\rho^{AB}) \leq E_c(\rho^{AB}) \leq E_f(\rho^{AB})$$

Distillable entanglement and entanglement cost

Bounds on E_d and E_c:

$$E_r(\rho^{AB}) \geq E_d(\rho^{AB}) \geq \max\left\{S(\rho^{A}) - S(\rho^{AB}), S(\rho^{B}) - S(\rho^{AB})\right\}$$

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Application: consider maximally correlated state

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For $\sigma_{\mathrm{sep}}^{AB} = \sum_i \alpha_{ii} |ii\rangle\langle ii|$ it holds

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We have: $S(\rho_{\mathrm{mc}}^{AB} \| \sigma_{\mathrm{sep}}^{AB}) \geq E_r(\rho^{AB}) \geq E_d(\rho^{AB}) \geq S(\rho_{\mathrm{mc}}^{A}) - S(\rho_{\mathrm{mc}}^{AB})$

$$\Rightarrow E_d(\rho_{\mathrm{mc}}^{AB}) = S(\rho_{\mathrm{mc}}^{A}) - S(\rho_{\mathrm{mc}}^{AB})$$

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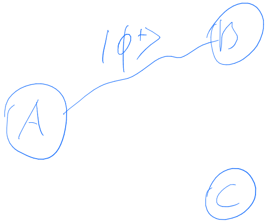
Exercise: If two qubits A and B are in the state $|\Phi^+\rangle$, prove that they cannot be correlated with another qubit C



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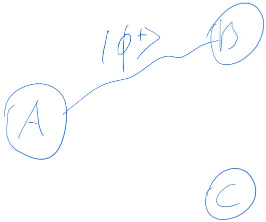

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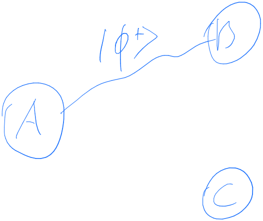

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- Reduced state is $\rho^{AB} = |\Phi^+\rangle\langle\Phi^+|^{AB}$

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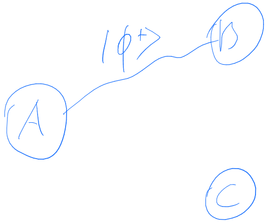


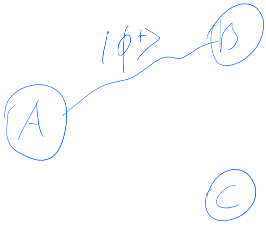
Solution:

- Consider total state ρ^{ABC}
- Reduced state is $\rho^{AB} = |\Phi^+\rangle\langle\Phi^+|^{AB}$
- $\Rightarrow$ total state must be

$$\rho^{ABC} = |\Phi^+\rangle\langle\Phi^+|^{AB} \otimes \rho^{C}$$

Entanglement monogamy

Entanglement monogamy: If two qubits *A* and *B* are maximally entangled, they cannot be entangled with another qubit *C*



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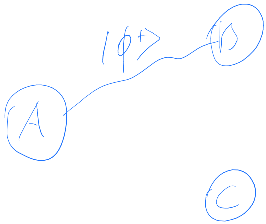

Compare to classical random variables: a classical random variable A can be maximally correlated with B and C at the same time:

$$\rho^{ABC} = \frac{1}{2}\,|000\rangle\langle 000| + \frac{1}{2}\,|111\rangle\langle 111|$$

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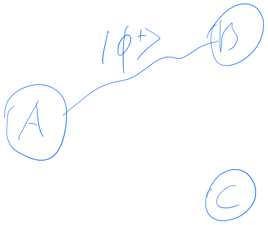

- For a pure state $|\psi\rangle^{ABC}$ it holds

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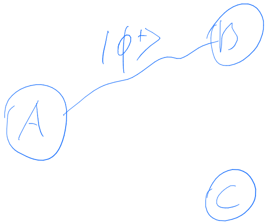

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- $C_{A:B}$ and $C_{A:C}$: concurrence of the reduced state ρ^{AB} and ρ^{AC}

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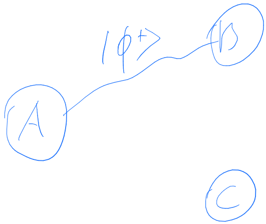

- For a pure state $|\psi\rangle^{ABC}$ it holds

$$C_{A:B}^2 + C_{A:C}^2 \leq C_{A:BC}^2$$

- $C_{A:B}$ and $C_{A:C}$: concurrence of the reduced state ρ^{AB} and ρ^{AC}
- $C_{A:BC} = \sqrt{2\left(1 - \mathrm{Tr}\left[(\rho^A)^2\right]\right)}$