Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

4th class
March 23, 2022

Advanced quantum information

- Every Wednesday 15:15 – 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al.*, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- 2. Homework sheet to be submitted via email by 5. April

Outline

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1. Entanglement distillation and dilution
   - Typical sequences
   - Entanglement dilution
   - Entanglement distillation
   - LOCC and separable operations
   - Mixed state entanglement distillation

Outline

Typical sequences

$P(\text{heads}) = 2/3$ $\qquad$ $P(\text{tails}) = 1/3$

- Consider a sequence $x_1, x_2, \ldots, x_m$ coming from m flips of a biased coin ($0 = \text{heads}$, $1 = \text{tails}$)

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- **Typical sequences:** sequences that are most likely to appear for large m

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ϵ**-typical sequence:** sequence of independent and identically distributed random variables x_i such that

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Exercise: For $\epsilon = 0.01$ and $m = 10$ is the sequence $1, 1, \ldots, 1, 1$ ϵ-typical?

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- $\Rightarrow 1, 1, \ldots, 1, 1$ is not ϵ-typical

Typical sequences

Theorem of typical sequences:
(1) Fix $\epsilon > 0$. For any $\delta > 0$, for sufficiently large m the probability that a sequence is ϵ-typical is at least $1 - \delta$:

$$\sum_{x\ \epsilon-\text{typical}} p(x_1)p(x_2)\ldots p(x_m) > 1 - \delta.$$

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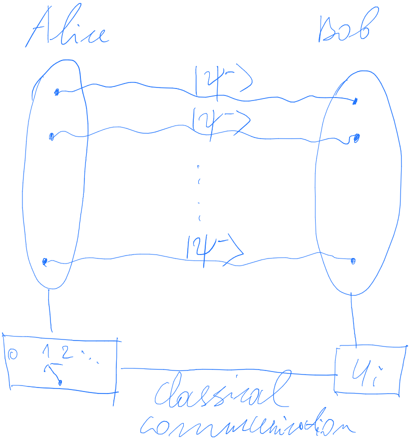
(2) For any fixed $\epsilon > 0$ and $\delta > 0$, for sufficiently large m, the number $|T(m, \epsilon)|$ of ϵ-typical sequences satisfies

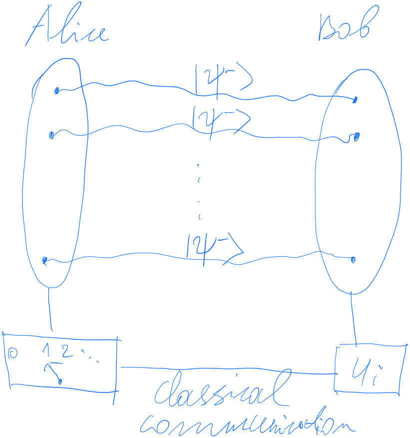
$$(1 - \delta)2^{m(H(p(x))-\epsilon)} \le |T(m, \epsilon)| \le 2^{m(H(p(x))+\epsilon)}.$$

Outline

Entanglement dilution

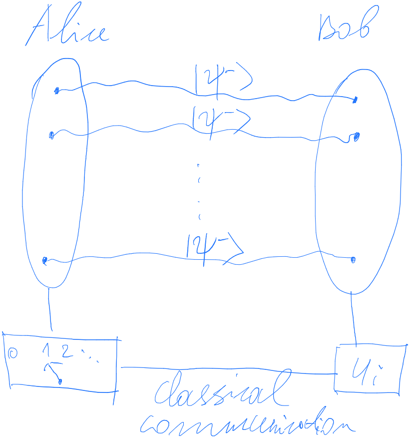


Entanglement dilution



Entanglement dilution: LOCC protocol transforming n singlets into m copies of $|\psi\rangle$

Entanglement dilution



Entanglement dilution: LOCC protocol transforming n singlets into m copies of $|\psi\rangle$

Entanglement cost of $|\psi\rangle$: minimal fraction $\frac{n}{m}$ in the limit $n \to \infty$

Entanglement dilution

Proposition 5.1. The entanglement cost of a state $|\psi\rangle$ is at most $S(\rho_\psi)$, where $\rho_\psi = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$ is the reduced state of Alice.

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The state $|\psi_m\rangle := |\psi\rangle^{\otimes m}$ can be written as

$$|\psi_m\rangle = \sum_{x_1, x_2, \ldots, x_m} \sqrt{p(x_1)p(x_2)\ldots p(x_m)}\, |x_1 x_2 \ldots x_m\rangle^A \otimes |x_1 x_2 \ldots x_m\rangle^B .$$

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Define $|\phi_m\rangle$ by omitting terms $x_1, \ldots, x_m$ which are not ϵ-typical:

$$|\phi_m\rangle = \sum_{x\ \epsilon-\text{typical}} \sqrt{p(x_1)p(x_2)\ldots p(x_m)}\, |x_1 x_2 \ldots x_m\rangle^A \otimes |x_1 x_2 \ldots x_m\rangle^B .$$

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Consider the scalar product $\langle\psi_m|\phi'_m\rangle$:

$$\begin{aligned}\langle\psi_m|\phi'_m\rangle &= \frac{1}{\sqrt{\langle\phi_m|\phi_m\rangle}} \sum_{x\ \epsilon\text{−typical}} p(x_1)p(x_2)\dots p(x_m) \\ &= \sqrt{\sum_{x\ \epsilon\text{−typical}} p(x_1)p(x_2)\dots p(x_m)}\end{aligned}$$

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Part (1) of the theorem of typical sequences implies that

$$\lim_{m\to\infty} \left(\sum_{x \ \epsilon-\text{typical}} p(x_1)p(x_2)\dots p(x_m) \right) = 1,$$

and thus

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$$\lim_{m\to\infty}\langle \psi_m | \phi'_m \rangle = 1.$$

$\Rightarrow$ $|\phi'_m\rangle$ **is a good approximation of** $|\psi\rangle^{\otimes m}$ **in the limit** $m \to \infty$

Entanglement dilution

- Alice prepares the state $|\phi'_m\rangle$ locally, and teleports half of it over to Bob

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- Part (2) of the theorem of typical sequences $\Rightarrow$ number of nonzero Schmidt coefficients of

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(and thus $|\phi'_m\rangle$) is at most $2^{m(H(p(x))+\epsilon)} = 2^{m(S(\rho_\psi)+\epsilon)}$

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- Recall Proposition 4.1.: for a state with k nonzero Schmidt coefficients teleportation can be done by consuming $\lceil \log_2 k \rceil$ Bell states
- $\Rightarrow$ Teleportation of $|\phi'_m\rangle$ can be performed by using at most

$$n = \left\lceil m(S(\rho_\psi) + \epsilon) \right\rceil$$

Bell states

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- $\Rightarrow$ Entanglement cost of $|\psi\rangle$ is at most $S(\rho_\psi)$

Q.E.D.

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Exercise: Estimate the entanglement cost for the state $|\psi\rangle = \sqrt{\frac{1}{3}}\,|00\rangle + \sqrt{\frac{2}{3}}\,|11\rangle$. Can entanglement cost be larger than 1?

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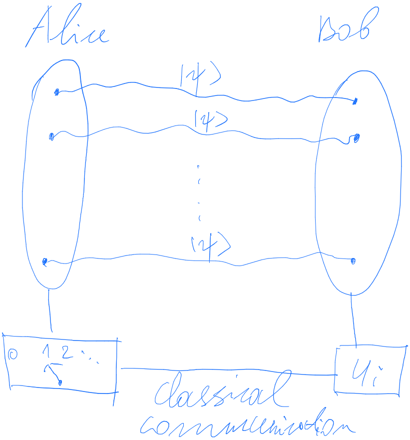
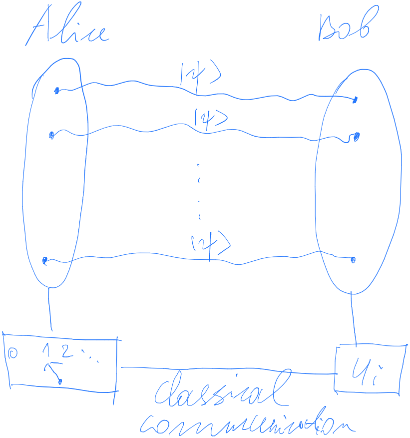
Entanglement distillation

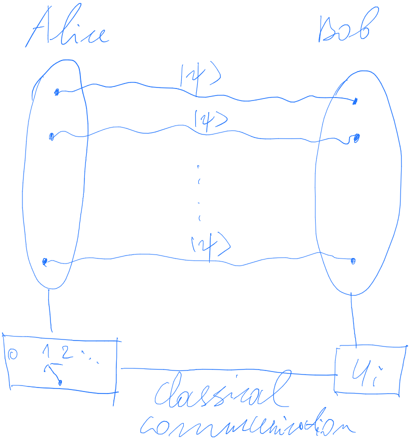


Entanglement distillation



Entanglement distillation: reverse of entanglement dilution, LOCC protocol converting m copies of $|\psi\rangle$ into n singlets

Entanglement distillation



Entanglement distillation: reverse of entanglement dilution, LOCC protocol converting m copies of $|\psi\rangle$ into n singlets

Distillable entanglement of $|\psi\rangle$: maximal fraction $\frac{n}{m}$ in the limit $m \to \infty$

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Alice performs a projective measurement with Kraus operators

$$\Pi_0 = \sum_{x\ \epsilon-\text{typical}} |x_1 x_2 \ldots x_m\rangle\langle x_1 x_2 \ldots x_m|$$

and $\Pi_1 = \mathbb{1} - \Pi_0$.

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Exercise: evaluate $p_0 = \mathrm{Tr}[(\Pi_0 \otimes \mathbb{1})\,|\psi\rangle\langle\psi|^{\otimes m}]$

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Entanglement distillation

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Post-measurement state of Alice and Bob:

$$|\phi'_m\rangle = \frac{1}{\sqrt{p_0}} \left(\Pi_0 \otimes \mathbb{1}\right) |\psi\rangle^{\otimes m} =$$

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- By definition of ϵ-typical sequences:

$$p(x_1)p(x_2)\dots p(x_m) \leq 2^{-m(H(p(x))-\epsilon)} = 2^{-m(S(\rho_\psi)-\epsilon)}$$

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- Part (1) of theorem of typical sequences: $p_0 > 1 - \delta$ for any $\delta > 0$ and m large enough
- $\Rightarrow$ largest Schmidt coefficient of $|\phi'_m\rangle$ is at most

$$\frac{2^{-m(S(\rho_\psi)-\epsilon)}}{1 - \delta}$$

Entanglement distillation

Largest Schmidt coefficient of $|\phi'_m\rangle$ is at most

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Choose n such that

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- $\Rightarrow$ all eigenvalues of $\rho_{\phi'_m}$ are at most 2^{-n}

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- Schmidt coefficients of $|\phi'_m\rangle$ correspond to eigenvalues of $\rho_{\phi'_m}$
- $\Rightarrow$ all eigenvalues of $\rho_{\phi'_m}$ are at most 2^{-n}
- $\Rightarrow \vec{\lambda}_{\phi'_m}$ is majorized by the vector

$$\vec{v} = (\underbrace{2^{-n}, 2^{-n}, \ldots, 2^{-n}}_{2^n \text{ times}}, 0, \ldots, 0)$$

Entanglement distillation

Largest Schmidt coefficient of $|\phi'_m\rangle$ is at most

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- $\Rightarrow \vec{\lambda}_{\phi'_m}$ is majorized by the vector

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Entanglement distillation

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- ϵ and δ can be chosen arbitrary small $\Rightarrow n/m$ arbitrary close to $S(\rho_\psi)$ in the limit of large m

Q.E.D.

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- Proof for entanglement cost by similar reasoning. Q.E.D.

Outline

LOCC and separable operations

- Any LOCC protocol is a **separable operation**:

$$\rho^{AB} \to \Lambda_{\mathrm{LOCC}}\left(\rho^{AB}\right) = \sum_i A_i \otimes B_i \rho^{AB} A_i^\dagger \otimes B_i^\dagger$$

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- Not every separable operation is an LOCC

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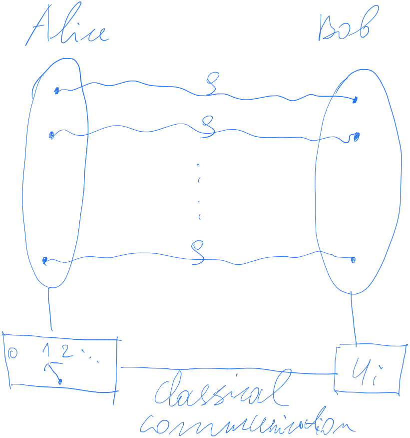
- Stochastic LOCC transformation mapping $\mathcal{H}_{AB}$ onto the space of two qubits: A_i is a $2 \times d_A$ rectangular matrix, B_i is a $2 \times d_B$ rectangular matrix

Outline

1. Entanglement distillation and dilution
   - Typical sequences
   - Entanglement dilution
   - Entanglement distillation
   - LOCC and separable operations
   - Mixed state entanglement distillation

Mixed state entanglement distillation



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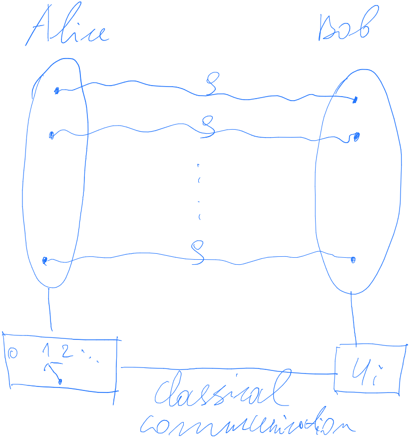

Entanglement distillation for **mixed states:** converting m copies of ρ into n singlets in the limit $m \to \infty$

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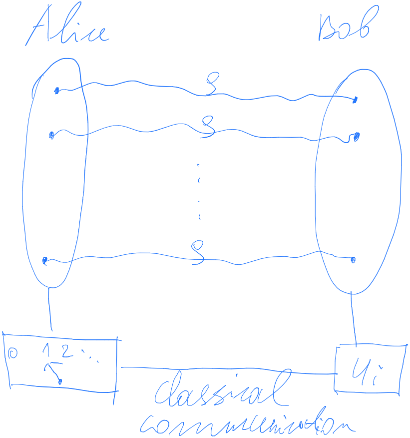

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Exercise: can a separable state $\rho_{\text{sep}} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$ be distilled into singlets?

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Separable states cannot be distilled into singlets:

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- Section 5.5.: stochastic LOCC brings $\rho^{\otimes m}$ to

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- If ρ is separable $\Rightarrow \rho^{\otimes m}$ is separable $\Rightarrow \sigma$ is separable