Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

5th class
March 30, 2022

Advanced quantum information

- Every Wednesday 15:15 – 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al.*, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- 2. Homework sheet to be submitted via email by 5. April

Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

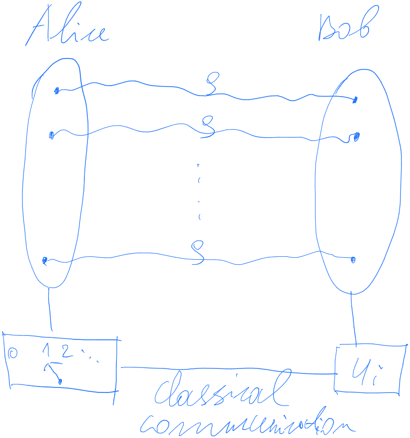
Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

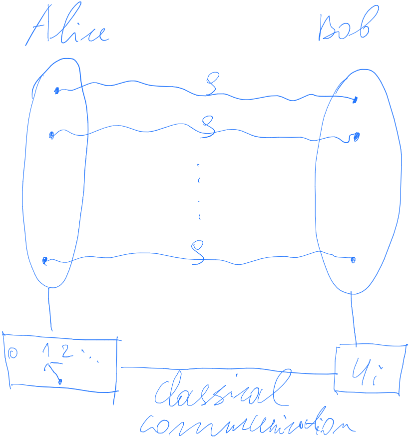
Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

Mixed state entanglement distillation

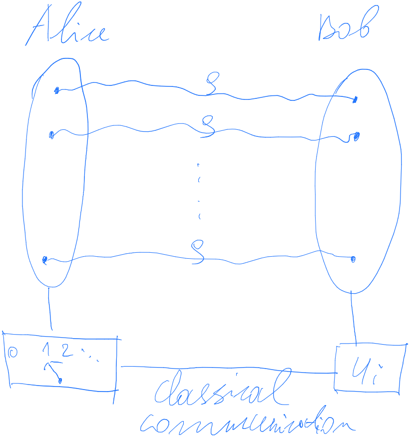


Mixed state entanglement distillation



Entanglement distillation for **mixed states:** converting m copies of ρ into n singlets in the limit $m \to \infty$

Mixed state entanglement distillation



Entanglement distillation for **mixed states:** converting m copies of ρ into n singlets in the limit $m \to \infty$

Exercise: can a separable state $\rho_{\text{sep}} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$ be distilled into singlets?

Mixed state entanglement distillation

Separable states cannot be distilled into singlets:

Mixed state entanglement distillation

Separable states cannot be distilled into singlets:

- Section 5.5.: stochastic LOCC brings $\rho^{\otimes m}$ to

$$\sigma = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

with probability $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$

Mixed state entanglement distillation

Separable states cannot be distilled into singlets:

- Section 5.5.: stochastic LOCC brings $\rho^{\otimes m}$ to

$$\sigma = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

  with probability $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$

- If ρ is separable $\Rightarrow \rho^{\otimes m}$ is separable $\Rightarrow \sigma$ is separable

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Assume ρ can be distilled into singlets

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Assume ρ can be distilled into singlets
- $\Rightarrow$ There exists a stochastic LOCC protocol bringing $\rho^{\otimes m}$ arbitrary close to a singlet for large m

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Assume ρ can be distilled into singlets
- $\Rightarrow$ There exists a stochastic LOCC protocol bringing $\rho^{\otimes m}$ arbitrary close to a singlet for large m
- There must exist a stochastic LOCC protocol transforming $\rho^{\otimes m}$ into an entangled two-qubit state σ_{2q}

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Section 5.5.: stochastic LOCC transformation has the form

$$\sigma_{2q} = \frac{1}{p} \sum_{j} A_j \otimes B_j \rho^{\otimes m} A_j^{\dagger} \otimes B_j^{\dagger}$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Section 5.5.: stochastic LOCC transformation has the form

$$\sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

- Probability: $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Section 5.5.: stochastic LOCC transformation has the form

$$\sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

- Probability: $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$
- A_j and B_j: $2 \times d_A$ and $2 \times d_B$ rectangular matrices

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled
- $\Rightarrow \sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p}\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled
- $\Rightarrow \sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- Probability: $p_i = \mathrm{Tr}[A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger]$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p}\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled
- $\Rightarrow \sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- Probability: $p_i = \mathrm{Tr}[A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger]$
- **Exercise:** prove that for entangled state σ_{2q} there must exist i such that σ_i is entangled

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled

- $\Rightarrow \sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i

- Probability: $p_i = \mathrm{Tr}[A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger]$

- **Exercise:** prove that for entangled state σ_{2q} there must exist i such that σ_i is entangled

- **Solution:** note that $\sigma_{2q} = \frac{1}{\sum_j p_j} \sum_i p_i \sigma_i$
$\Rightarrow \sigma_{2q}$ is separable if all σ_i are separable

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- $\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- A_i and B_i: rectangular $2 \times d_A$ and $2 \times d_B$ matrices

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- $\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- A_i and B_i: rectangular $2 \times d_A$ and $2 \times d_B$ matrices
- It follows:

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- $\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- A_i and B_i: rectangular $2 \times d_A$ and $2 \times d_B$ matrices
- It follows:

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

- $|\alpha_i\rangle \in \mathcal{H}_A$ and $|\beta_i\rangle \in \mathcal{H}_B$ are (possibly unnormalized) vectors

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$
- P_B: projector onto the subspace spanned by $|\beta_0\rangle$ and $|\beta_1\rangle$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$
- P_B: projector onto the subspace spanned by $|\beta_0\rangle$ and $|\beta_1\rangle$
- It holds

$$\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$$
$$= \frac{1}{p_i} A_i \otimes B_i \left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B \right) A_i^\dagger \otimes B_i^\dagger$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- It holds

$$\begin{aligned}\sigma_i &= \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger \\ &= \frac{1}{p_i} A_i \otimes B_i \Big(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B \Big) A_i^\dagger \otimes B_i^\dagger\end{aligned}$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- It holds

$$\begin{aligned}\sigma_i &= \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger \\ &= \frac{1}{p_i} A_i \otimes B_i \left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B \right) A_i^\dagger \otimes B_i^\dagger\end{aligned}$$

- σ_i is entangled $\Rightarrow$

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}$$

  must be entangled

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}$$

- Consider orthonormal product basis $|f_i\rangle \otimes |g_k\rangle$ such that

$$P_A = |f_0\rangle\langle f_0| + |f_1\rangle\langle f_1|$$
$$P_B = |g_0\rangle\langle g_0| + |g_1\rangle\langle g_1|$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}$$

- Consider orthonormal product basis $|f_i\rangle \otimes |g_k\rangle$ such that

$$\begin{aligned} P_A &= |f_0\rangle\langle f_0| + |f_1\rangle\langle f_1| \\ P_B &= |g_0\rangle\langle g_0| + |g_1\rangle\langle g_1| \end{aligned}$$

- In the basis $|f_i\rangle \otimes |g_k\rangle$ the state μ takes the form

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$
- If $\tau_{2q}^{T_A}$ is positive $\Rightarrow \tau_{2q}$ is separable (Theorem 3.2.)

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$
- If $\tau_{2q}^{T_A}$ is positive $\Rightarrow$ τ_{2q} is separable (Theorem 3.2.)
- $\Rightarrow$ **Contradiction:** μ is separable but we showed previously that μ must be entangled!

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$
- If $\tau_{2q}^{T_A}$ is positive $\Rightarrow \tau_{2q}$ is separable (Theorem 3.2.)
- $\Rightarrow$ **Contradiction:** μ is separable but we showed previously that μ must be entangled!
- $\Rightarrow \tau_{2q}^{T_A}$ must have negative eigenvalues

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

There exists a vector

$$|\psi\rangle = \sum_{i,k=0}^{1} c_{ik} |f_i\rangle |g_k\rangle$$

such that $\langle\psi|\tau_{2q}^{T_A}|\psi\rangle < 0$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

We have $\langle\psi|\tau_{2q}^{T_A}|\psi\rangle = \langle\psi|\mu^{T_A}|\psi\rangle$, which implies that

$$\langle\psi|\mu^{T_A}|\psi\rangle < 0$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}, \qquad \langle\psi|\mu^{T_A}|\psi\rangle < 0$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}, \qquad \langle\psi|\mu^{T_A}|\psi\rangle < 0$$

The following equalities hold:

$$\left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right)^{T_A} = P_A \otimes P_B \left(\rho^{\otimes m}\right)^{T_A} P_A \otimes P_B,$$
$$P_A \otimes P_B \left|\psi\right\rangle = \left|\psi\right\rangle$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\operatorname{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}, \qquad \langle\psi|\mu^{T_A}|\psi\rangle < 0$$

The following equalities hold:

$$\left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right)^{T_A} = P_A \otimes P_B \left(\rho^{\otimes m}\right)^{T_A} P_A \otimes P_B,$$
$$P_A \otimes P_B \left|\psi\right\rangle = \left|\psi\right\rangle$$

We obtain:

$$0 > \langle\psi|\mu^{T_A}|\psi\rangle = \frac{\langle\psi|\left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right)^{T_A}|\psi\rangle}{\operatorname{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} =$$
$$= \frac{\langle\psi|P_A \otimes P_B \left(\rho^{\otimes m}\right)^{T_A} P_A \otimes P_B|\psi\rangle}{\operatorname{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \frac{\langle\psi|\left(\rho^{\otimes m}\right)^{T_A}|\psi\rangle}{\operatorname{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Exercise: prove that ρ^{T_A} is not positive semidefinite

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Exercise: prove that ρ^{T_A} is not positive semidefinite

Solution:

- for two matrices $M_1^{A_1 B_1}$ and $M_2^{A_2 B_2}$ it holds that

$$\left(M_1^{A_1 B_1} \otimes M_2^{A_2 B_2}\right)^{T_{A_1 A_2}} = \left(M_1^{A_1 B_1}\right)^{T_{A_1}} \otimes \left(M_2^{A_2 B_2}\right)^{T_{A_2}},$$

  and similar for more than 2 matrices

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Exercise: prove that ρ^{T_A} is not positive semidefinite

Solution:

- for two matrices $M_1^{A_1 B_1}$ and $M_2^{A_2 B_2}$ it holds that

$$\left(M_1^{A_1 B_1} \otimes M_2^{A_2 B_2}\right)^{T_{A_1 A_2}} = \left(M_1^{A_1 B_1}\right)^{T_{A_1}} \otimes \left(M_2^{A_2 B_2}\right)^{T_{A_2}},$$

  and similar for more than 2 matrices
- $\Rightarrow \left(\rho^{\otimes m}\right)^{T_A} = \left(\rho^{T_A}\right)^{\otimes m}$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Exercise: prove that ρ^{T_A} is not positive semidefinite

Solution:

- for two matrices $M_1^{A_1 B_1}$ and $M_2^{A_2 B_2}$ it holds that

$$\left(M_1^{A_1 B_1} \otimes M_2^{A_2 B_2}\right)^{T_{A_1 A_2}} = \left(M_1^{A_1 B_1}\right)^{T_{A_1}} \otimes \left(M_2^{A_2 B_2}\right)^{T_{A_2}},$$

  and similar for more than 2 matrices
- $\Rightarrow \left(\rho^{\otimes m}\right)^{T_A} = \left(\rho^{T_A}\right)^{\otimes m}$
- $\Rightarrow \rho^{T_A}$ must have negative eigenvalues
  Q.E.D.

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled
- **Are there entangled states which cannot be distilled?**

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled
- **Are there entangled states which cannot be distilled?**
- Independent entanglement detection criterion required

Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

Matrix realignment criterion

- Consider a 2×2 matrix

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

Matrix realignment criterion

- Consider a 2×2 matrix

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

- We can "vectorize" M by defining

$$\vec{M} = (M_{00}, M_{10}, M_{01}, M_{11})^T$$

Matrix realignment criterion

- Consider a 2×2 matrix

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

- We can "vectorize" M by defining

$$\vec{M} = (M_{00}, M_{10}, M_{01}, M_{11})^T$$

- Similar for larger dimensions

Matrix realignment criterion

- Consider two-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^{\dagger} & Z \end{pmatrix}$$

Matrix realignment criterion

- Consider two-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

- **Realigned matrix**

$$\tilde{\rho} = \begin{pmatrix} \vec{X}^T \\ \overrightarrow{Y^\dagger}^T \\ \vec{Y}^T \\ \vec{Z}^T \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{10} & \rho_{01} & \rho_{11} \\ \rho_{20} & \rho_{30} & \rho_{21} & \rho_{31} \\ \rho_{02} & \rho_{12} & \rho_{03} & \rho_{13} \\ \rho_{22} & \rho_{32} & \rho_{23} & \rho_{33} \end{pmatrix}$$

Matrix realignment criterion

- Consider two-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

- **Realigned matrix**

$$\tilde{\rho} = \begin{pmatrix} \vec{X}^T \\ \overrightarrow{Y^\dagger}^T \\ \vec{Y}^T \\ \vec{Z}^T \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{10} & \rho_{01} & \rho_{11} \\ \rho_{20} & \rho_{30} & \rho_{21} & \rho_{31} \\ \rho_{02} & \rho_{12} & \rho_{03} & \rho_{13} \\ \rho_{22} & \rho_{32} & \rho_{23} & \rho_{33} \end{pmatrix}$$

- Similar for larger dimensions

Matrix realignment criterion

- **Trace norm** of $\tilde{\rho}$ can be used to detect entanglement

Matrix realignment criterion

- **Trace norm** of $\tilde{\rho}$ can be used to detect entanglement
- Trace norm of M:

$$||M||_1 = \text{Tr}\,\sqrt{M^\dagger M} = \sum_i s_i$$

Matrix realignment criterion

- **Trace norm** of $\tilde{\rho}$ can be used to detect entanglement
- Trace norm of M:

$$\|M\|_1 = \mathrm{Tr}\,\sqrt{M^\dagger M} = \sum_i s_i$$

- Triangle inequality:

$$\|A + B\|_1 \leq \|A\|_1 + \|B\|_1$$

Matrix realignment criterion

- **Trace norm** of $\tilde{\rho}$ can be used to detect entanglement
- Trace norm of M:
$$\|M\|_1 = \mathrm{Tr}\,\sqrt{M^\dagger M} = \sum_i s_i$$
- Triangle inequality:
$$\|A + B\|_1 \leq \|A\|_1 + \|B\|_1$$
- Trace norm is absolutely homogeneous:
$$\|aM\|_1 = |a| \cdot \|M\|_1$$

  for any matrix M and any $a \in \mathbb{C}$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Let ρ be a pure product state:

$$\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Let ρ be a pure product state:

$$\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$$

- "Vectorize" the matrices $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ with the corresponding vectors $\vec{\psi}$ and $\vec{\phi}$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Let ρ be a pure product state:

$$\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$$

- "Vectorize" the matrices $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ with the corresponding vectors $\vec{\psi}$ and $\vec{\phi}$

- It holds

$$|\vec{\psi}| = |\vec{\phi}| = 1$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Let ρ be a pure product state:

$$\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$$

- "Vectorize" the matrices $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ with the corresponding vectors $\vec{\psi}$ and $\vec{\phi}$

- It holds

$$|\vec{\psi}| = |\vec{\phi}| = 1$$

- Realigned matrix $\tilde{\rho}$ can be written as

$$\tilde{\rho} = \vec{\psi} \cdot \vec{\phi}^T$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Let ρ be a pure product state:

$$\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$$

- "Vectorize" the matrices $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ with the corresponding vectors $\vec{\psi}$ and $\vec{\phi}$

- It holds

$$|\vec{\psi}| = |\vec{\phi}| = 1$$

- Realigned matrix $\tilde{\rho}$ can be written as

$$\tilde{\rho} = \vec{\psi} \cdot \vec{\phi}^T$$

- Trace norm of $\tilde{\rho}$ is given as $||\tilde{\rho}||_1 = 1$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $\|\tilde{\rho}\|_1 \leq 1$.

Proof.

- Consider a separable state

$$\rho_{\mathrm{sep}} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Consider a separable state

$$\rho_{\text{sep}} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

- Realigned matrix $\widetilde{\rho_{\text{sep}}}$ takes the form

$$\widetilde{\rho_{\text{sep}}} = \sum_i p_i \overrightarrow{\psi}_i \cdot \overrightarrow{\phi}_i^T$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

- Consider a separable state

$$\rho_{\text{sep}} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

- Realigned matrix $\widetilde{\rho_{\text{sep}}}$ takes the form

$$\widetilde{\rho_{\text{sep}}} = \sum_i p_i \overrightarrow{\psi}_i \cdot \overrightarrow{\phi}_i^{\,T}$$

- $\overrightarrow{\psi}_i$ and $\overrightarrow{\phi}_i$: "vectorized" matrices $|\psi_i\rangle\langle\psi_i|$ and $|\phi_i\rangle\langle\phi_i|$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

Proof.

$$\widetilde{\rho_{\text{sep}}} = \sum_i p_i \overrightarrow{\psi}_i \cdot \overrightarrow{\phi}_i^T$$

Matrix realignment criterion

Proposition 5.3. Any separable state ρ fulfills $\|\tilde{\rho}\|_1 \leq 1$.

Proof.

$$\widetilde{\rho_{\text{sep}}} = \sum_i p_i \overrightarrow{\psi_i} \cdot \overrightarrow{\phi_i}^T$$

Trace norm of $\widetilde{\rho_{\text{sep}}}$:

$$\|\widetilde{\rho_{\text{sep}}}\|_1 = \left\| \sum_i p_i \overrightarrow{\psi_i} \cdot \overrightarrow{\phi_i}^T \right\|_1 \leq \sum_i p_i \left\| \overrightarrow{\psi_i} \cdot \overrightarrow{\phi_i}^T \right\|_1 = 1$$

Q.E.D.

Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

Bound entanglement

For $d_A = d_B = 3$ and $0 \le a \le 1$ consider

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

Bound entanglement

For $d_A = d_B = 3$ and $0 \leq a \leq 1$ consider

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

- ρ_a is PPT for all $0 \leq a \leq 1$

Bound entanglement

For $d_A = d_B = 3$ and $0 \leq a \leq 1$ consider

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

- ρ_a is PPT for all $0 \leq a \leq 1$
- $||\tilde{\rho}||_1 > 1$ for all $0 < a < 1$

Bound entanglement

For $d_A = d_B = 3$ and $0 \leq a \leq 1$ consider

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

- ρ_a is PPT for all $0 \leq a \leq 1$
- $||\tilde{\rho}||_1 > 1$ for all $0 < a < 1$
- $\Rightarrow \rho_a$ is bound entangled for $0 < a < 1$

Bound entanglement

In summary:

- All separable states and some entangled states have positive partial transpose (PPT)

Bound entanglement

In summary:

- All separable states and some entangled states have positive partial transpose (PPT)
- PPT states cannot be distilled into singlets

Bound entanglement

In summary:

- All separable states and some entangled states have positive partial transpose (PPT)

- PPT states cannot be distilled into singlets

- There are PPT entangled states $\Rightarrow$ these states require singlets to be created, but cannot be converted into singlets

Bound entanglement

In summary:

- All separable states and some entangled states have positive partial transpose (PPT)
- PPT states cannot be distilled into singlets
- There are PPT entangled states ⇒ these states require singlets to be created, but cannot be converted into singlets
- These states are called **bound entangled**

Outline

Entanglement quantification

How much entanglement is in a given quantum state ρ?

Entanglement quantification

How much entanglement is in a given quantum state ρ?

Entanglement measure: function $E(\rho)$ with following properties

Entanglement quantification

How much entanglement is in a given quantum state ρ?

Entanglement measure: function $E(\rho)$ with following properties

1. $E(\rho) \geq 0$, and equality holds if ρ is separable,

Entanglement quantification

How much entanglement is in a given quantum state ρ?

Entanglement measure: function $E(\rho)$ with following properties

1. $E(\rho) \geq 0$, and equality holds if ρ is separable,
2. E does not increase under local operations and classical communication:

   $$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho)$$

   for any LOCC protocol Λ_{LOCC}

Entanglement quantification

How much entanglement is in a given quantum state ρ?

Entanglement measure: function $E(\rho)$ with following properties

1. $E(\rho) \geq 0$, and equality holds if ρ is separable,
2. E does not increase under local operations and classical communication:
$$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho)$$
for any LOCC protocol Λ_{LOCC}

Theorem 2.1.: $|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ can be converted into any other state ρ via LOCC $\Rightarrow |\Phi_d^+\rangle$ has maximum entanglement

$$E(\rho) = E\left(\Lambda_{\mathrm{LOCC}}\left[|\Phi_d^+\rangle\langle\Phi_d^+|\right]\right) \leq E(|\Phi_d^+\rangle)$$

Outline

Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

where $\rho^A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

where $\rho^A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

- For mixed states:

$$E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

where $\rho^A = \text{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

- For mixed states:

$$E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

- Minimum is taken over all decompositions $\{p_i, |\psi_i\rangle^{AB}\}$ suth that $\rho^{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{AB}$

Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

where $\rho^A = \text{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

- For mixed states:

$$E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

- Minimum is taken over all decompositions $\{p_i, |\psi_i\rangle^{AB}\}$ suth that $\rho^{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{AB}$

- **Interpretation:** minimal average entanglement required to create ρ^{AB}

Entanglement of formation

$$E_f(|\psi\rangle^{AB}) = S(\rho^A), \quad E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

Entanglement of formation

$$E_f(|\psi\rangle^{AB}) = S(\rho^A), \quad E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

Exercise: prove that $E_f(\rho^{AB}) \geq 0$, and $E_f(\sigma^{AB}) = 0$ for any separable state σ^{AB}

Entanglement of formation

$$E_f(|\psi\rangle^{AB}) = S(\rho^A), \quad E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

Exercise: prove that $E_f(\rho^{AB}) \geq 0$, and $E_f(\sigma^{AB}) = 0$ for any separable state σ^{AB}

Solution:

- For any decomposition $\{p_i, |\psi_i\rangle^{AB}\}$ the average entanglement $\sum_i p_i E_f(|\psi_i\rangle^{AB})$ is nonnegative

Entanglement of formation

$$E_f(|\psi\rangle^{AB}) = S(\rho^A), \quad E_f(\rho^{AB}) = \min \sum_i p_i E_f(|\psi_i\rangle^{AB})$$

Exercise: prove that $E_f(\rho^{AB}) \geq 0$, and $E_f(\sigma^{AB}) = 0$ for any separable state σ^{AB}

Solution:

- For any decomposition $\{p_i, |\psi_i\rangle^{AB}\}$ the average entanglement $\sum_i p_i E_f(|\psi_i\rangle^{AB})$ is nonnegative
- For a separable state σ^{AB} there exists a decompotision into product states $|\psi_i\rangle^{AB} = |\alpha_i\rangle^A \otimes |\beta_i\rangle^B$ with $E_f(|\psi_i\rangle^{AB}) = 0$

Entanglement of formation

Next goal: proving that E_f does not increase under LOCC

Entanglement of formation

Next goal: proving that E_f does not increase under LOCC

For this we will prove that:

- E_f is convex

Entanglement of formation

Next goal: proving that E_f does not increase under LOCC

For this we will prove that:

- E_f is convex
- E_f does not increase on average under local measurements for
  - pure states

Entanglement of formation

Next goal: proving that E_f does not increase under LOCC

For this we will prove that:

- E_f is convex
- E_f does not increase on average under local measurements for
  - pure states
  - mixed states

Entanglement of formation

Next goal: proving that E_f does not increase under LOCC

For this we will prove that:

- E_f is convex
- E_f does not increase on average under local measurements for
  - pure states
  - mixed states

$\Rightarrow$ in combination, this will prove that E_f does not increase under LOCC

Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i\rho_i^{AB}\right) \le \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i\rho_i^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Proof.

- Consider a decomposition of $\rho_i^{AB} = \sum_j q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$ with the property that

$$E_f(\rho_i^{AB}) = \sum_j q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right).$$

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i\rho_i^{AB}\right) \le \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Proof.

- Consider a decomposition of $\rho_i^{AB} = \sum_j q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$ with the property that

$$E_f(\rho_i^{AB}) = \sum_j q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right).$$

- Defining $\sigma^{AB} = \sum_i p_i\rho_i^{AB}$ we obtain

$$\sigma^{AB} = \sum_i p_i\rho_i^{AB} = \sum_{ij} p_i q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$$

$$\sum_i p_i E_f\left(\rho_i^{AB}\right) = \sum_{ij} p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i\rho_i^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Proof.

$$\sigma^{AB} = \sum_i p_i\rho_i^{AB} = \sum_{ij} p_i q_{ij} \left|\psi_{ij}\right\rangle\!\left\langle\psi_{ij}\right|^{AB}$$

$$\sum_i p_i E_f\left(\rho_i^{AB}\right) = \sum_{ij} p_i q_{ij} E_f\left(\left|\psi_{ij}\right\rangle^{AB}\right)$$

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i \rho_i^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Proof.

$$\sigma^{AB} = \sum_i p_i \rho_i^{AB} = \sum_{ij} p_i q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$$

$$\sum_i p_i E_f\left(\rho_i^{AB}\right) = \sum_{ij} p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

- E_f is the minimal average entanglement $\Rightarrow$

$$E_f\left(\sigma^{AB}\right) \leq \sum_i p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

Convexity of E_f

Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i \rho_i^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

Proof.

$$\sigma^{AB} = \sum_i p_i \rho_i^{AB} = \sum_{ij} p_i q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$$

$$\sum_i p_i E_f\left(\rho_i^{AB}\right) = \sum_{ij} p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

- E_f is the minimal average entanglement $\Rightarrow$

$$E_f\left(\sigma^{AB}\right) \leq \sum_i p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

- In summary: $E_f\left(\sum_i p_i \rho_i^{AB}\right) = E_f\left(\sigma^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right)$. Q.E.D.

Outline

Monotonicity of E_f under local measurements

Proposition 6.2. For pure states $|\psi\rangle^{AB}$ entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(|\phi_i\rangle^{AB}) \leq E_f(|\psi\rangle^{AB})$$

with

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1}\, |\psi\rangle\langle\psi|^{AB}\, K_i^\dagger \otimes \mathbb{1}\right],$$

$$|\phi_i\rangle^{AB} = \frac{1}{\sqrt{p_i}}(K_i \otimes \mathbb{1})\, |\psi\rangle^{AB}.$$

Entanglement of formation

Proof.

- Local measurements on Alice's side do not change the state of Bob

Entanglement of formation

Proof.

- Local measurements on Alice's side do not change the state of Bob

- Thus

$$\rho^B = \mathrm{Tr}_A\left[|\psi\rangle\langle\psi|^{AB}\right] = \sum_i p_i \,\mathrm{Tr}_A\left[|\phi_i\rangle\langle\phi_i|^{AB}\right] = \sum_i p_i \sigma_i^B$$

  with $\sigma_i^B = \mathrm{Tr}_A\left[|\phi_i\rangle\langle\phi_i|^{AB}\right]$

Entanglement of formation

Proof.

- Local measurements on Alice's side do not change the state of Bob

- Thus

$$\rho^B = \mathrm{Tr}_A\left[|\psi\rangle\langle\psi|^{AB}\right] = \sum_i p_i \,\mathrm{Tr}_A\left[|\phi_i\rangle\langle\phi_i|^{AB}\right] = \sum_i p_i \sigma_i^B$$

with $\sigma_i^B = \mathrm{Tr}_A\left[|\phi_i\rangle\langle\phi_i|^{AB}\right]$

- By definition of E_f we have

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B).$$

Entanglement of formation

Proof.

$$\rho^B = \sum_i p_i \sigma_i^B, \quad \sigma_i^B = \mathrm{Tr}_A \left[|\phi_i\rangle\langle\phi_i|^{AB} \right]$$

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B)$$

Entanglement of formation

Proof.

$$\rho^B = \sum_i p_i \sigma_i^B, \quad \sigma_i^B = \mathrm{Tr}_A \left[|\phi_i\rangle\langle\phi_i|^{AB} \right]$$

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B)$$

- von Neumann entropy is concave: $\sum_i p_i S(\sigma_i^B) \le S\left(\sum_i p_i \sigma_i^B\right)$

Entanglement of formation

Proof.

$$\rho^B = \sum_i p_i \sigma_i^B, \quad \sigma_i^B = \mathrm{Tr}_A \left[|\phi_i\rangle\langle\phi_i|^{AB} \right]$$

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B)$$

- von Neumann entropy is concave: $\sum_i p_i S(\sigma_i^B) \leq S\left(\sum_i p_i \sigma_i^B\right)$
- We have:

$$\begin{aligned}\sum_i p_i E_f(|\phi_i\rangle^{AB}) &= \sum_i p_i S(\sigma_i^B) \leq S\left(\sum_i p_i \sigma_i^B\right) \\ &= S(\rho^B) = E_f(|\psi\rangle^{AB})\end{aligned}$$

Entanglement of formation

Proof.

$$\rho^B = \sum_i p_i \sigma_i^B, \quad \sigma_i^B = \mathrm{Tr}_A \left[|\phi_i\rangle\langle\phi_i|^{AB} \right]$$

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B)$$

- von Neumann entropy is concave: $\sum_i p_i S(\sigma_i^B) \leq S\left(\sum_i p_i \sigma_i^B\right)$
- We have:

$$\sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B) \leq S\left(\sum_i p_i \sigma_i^B\right)$$
$$= S(\rho^B) = E_f(|\psi\rangle^{AB})$$

- In summary: $\sum_i p_i E_f(|\phi_i\rangle^{AB}) \leq E_f(|\psi\rangle^{AB})$
  Q.E.D.

Monotonicity of E_f under local measurements

Extension to mixed states ρ^{AB} and local Kraus operators K_i:

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1}\rho^{AB} K_i^\dagger \otimes \mathbb{1}\right]$$

$$\sigma_i^{AB} = \frac{1}{p_i} K_i \otimes \mathbb{1}\rho^{AB} K_i^\dagger \otimes \mathbb{1}$$

Monotonicity of E_f under local measurements

Extension to mixed states ρ^{AB} and local Kraus operators K_i:

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1} \rho^{AB} K_i^\dagger \otimes \mathbb{1}\right]$$
$$\sigma_i^{AB} = \frac{1}{p_i} K_i \otimes \mathbb{1} \rho^{AB} K_i^\dagger \otimes \mathbb{1}$$

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

- Consider optimal decomposition $\rho^{AB} = \sum_j q_j |\psi_j\rangle\langle\psi_j|^{AB}$ such that

$$E_f(\rho^{AB}) = \sum_j q_j E_f(|\psi_j\rangle^{AB})$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

- Consider optimal decomposition $\rho^{AB} = \sum_j q_j |\psi_j\rangle\langle\psi_j|^{AB}$ such that

$$E_f(\rho^{AB}) = \sum_j q_j E_f(|\psi_j\rangle^{AB})$$

- Define

$$p_{ij} = \mathrm{Tr}\left[(K_i \otimes \mathbb{1})\,|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$
$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\,(K_i \otimes \mathbb{1})\,|\psi_j\rangle^{AB}$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

- Consider optimal decomposition $\rho^{AB} = \sum_j q_j |\psi_j\rangle\langle\psi_j|^{AB}$ such that

$$E_f(\rho^{AB}) = \sum_j q_j E_f(|\psi_j\rangle^{AB})$$

- Define

$$p_{ij} = \operatorname{Tr}\left[(K_i \otimes \mathbb{1})\,|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$
$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\,(K_i \otimes \mathbb{1})\,|\psi_j\rangle^{AB}$$

- Note that $\sum_j q_j p_{ij} = p_i$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$p_{ij} = \mathrm{Tr}\left[(K_i \otimes \mathbb{1})\, |\psi_j\rangle\langle\psi_j| \left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$

$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\,(K_i \otimes \mathbb{1})\, |\psi_j\rangle^{AB}$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$p_{ij} = \mathrm{Tr}\left[\left(K_i \otimes \mathbb{1}\right) |\psi_j\rangle\langle\psi_j| \left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$

$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}} \left(K_i \otimes \mathbb{1}\right) |\psi_j\rangle^{AB}$$

For the entanglement of formation of σ_i^{AB} we obtain

$$\begin{aligned} E_f\left(\sigma_i^{AB}\right) &= E_f\left(\frac{1}{p_i} K_i \otimes \mathbb{1} \rho^{AB} K_i^\dagger \otimes \mathbb{1}\right) \\ &= E_f\left(\sum_j \frac{q_j}{p_i} K_i \otimes \mathbb{1}\, |\psi_j\rangle\langle\psi_j|^{AB}\, K_i^\dagger \otimes \mathbb{1}\right) \end{aligned}$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$p_{ij} = \mathrm{Tr}\left[(K_i \otimes \mathbb{1})\,|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$

$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\,(K_i \otimes \mathbb{1})\,|\psi_j\rangle^{AB}$$

For the entanglement of formation of σ_i^{AB} we obtain

$$E_f\left(\sigma_i^{AB}\right) = E_f\left(\sum_j \frac{q_j}{p_i} K_i \otimes \mathbb{1}\,|\psi_j\rangle\langle\psi_j|^{AB}\,K_i^\dagger \otimes \mathbb{1}\right)$$

$$= E_f\left(\sum_j \frac{q_j p_{ij}}{p_i}\,|\phi_{ij}\rangle\langle\phi_{ij}|^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$E_f\left(\sigma_i^{AB}\right) = E_f\left(\sum_j \frac{q_j p_{ij}}{p_i} \left|\phi_{ij}\right\rangle\!\left\langle\phi_{ij}\right|^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$E_f\left(\sigma_i^{AB}\right) = E_f\left(\sum_j \frac{q_j p_{ij}}{p_i} |\phi_{ij}\rangle\langle\phi_{ij}|^{AB}\right)$$

- Convexity of E_f:

$$E_f\left(\sigma_i^{AB}\right) \leq \sum_j \frac{q_j p_{ij}}{p_i} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$E_f\left(\sigma_i^{AB}\right) = E_f\left(\sum_j \frac{q_j p_{ij}}{p_i} |\phi_{ij}\rangle\langle\phi_{ij}|^{AB}\right)$$

- Convexity of E_f:

$$E_f\left(\sigma_i^{AB}\right) \leq \sum_j \frac{q_j p_{ij}}{p_i} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

- Leading to

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_{i,j} q_j p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_{i,j} q_j p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_{i,j} q_j p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

- Recall that

$$p_{ij} = \mathrm{Tr}\left[\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$
$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle^{AB}$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_{i,j} q_j p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

- Recall that

$$p_{ij} = \mathrm{Tr}\left[\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$
$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle^{AB}$$

- Proposition 6.2 $\Rightarrow \sum_i p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right) \leq E_f\left(|\psi_j\rangle^{AB}\right)$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_{i,j} q_j p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right)$$

- Recall that

$$p_{ij} = \mathrm{Tr}\left[\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle\langle\psi_j|\left(K_i^\dagger \otimes \mathbb{1}\right)\right]$$
$$|\phi_{ij}\rangle^{AB} = \frac{1}{\sqrt{p_{ij}}}\left(K_i \otimes \mathbb{1}\right)|\psi_j\rangle^{AB}$$

- Proposition 6.2 $\Rightarrow \sum_i p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right) \leq E_f\left(|\psi_j\rangle^{AB}\right)$
- $\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_j q_j \sum_i p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right) \leq \sum_j q_j E_f\left(|\psi_j\rangle^{AB}\right)$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_j q_j E_f\left(|\psi_j\rangle^{AB}\right)$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_j q_j E_f\left(|\psi_j\rangle^{AB}\right)$$

- Recall that $\{q_j, |\psi_j\rangle^{AB}\}$ is an optimal decomposition:

$$E_f(\rho^{AB}) = \sum_j q_j E_f(|\psi_j\rangle^{AB})$$

Monotonicity of E_f under local measurements

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side: $\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB})$

Proof.

$$\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_j q_j E_f\left(|\psi_j\rangle^{AB}\right)$$

- Recall that $\{q_j, |\psi_j\rangle^{AB}\}$ is an optimal decomposition:

$$E_f(\rho^{AB}) = \sum_j q_j E_f(|\psi_j\rangle^{AB})$$

- $\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq E_f\left(\rho^{AB}\right)$

Q.E.D.

Outline

1. Entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Quantification of entanglement
3. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits

Monotonicity of E_f under LOCC

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Monotonicity of E_f under LOCC

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Generalizes to local measurements on Alice's and Bob's side, with exchange of measurement outcomes via classical channel

Monotonicity of E_f under LOCC

Proposition 6.3. For all mixed states ρ^{AB} the entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Generalizes to local measurements on Alice's and Bob's side, with exchange of measurement outcomes via classical channel

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Monotonicity of E_f under LOCC

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Monotonicity of E_f under LOCC

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Theorem 6.1. Entanglement of formation does not increase under LOCC:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E_f(\rho)$$

for any LOCC protocol Λ_{LOCC}.

Monotonicity of E_f under LOCC

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Theorem 6.1. Entanglement of formation does not increase under LOCC:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E_f(\rho)$$

for any LOCC protocol Λ_{LOCC}.

Exercise: prove this theorem from Proposition 6.4. by using convexity of E_f

Monotonicity of E_f under LOCC

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Theorem 6.1. Entanglement of formation does not increase under LOCC:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E_f(\rho)$$

for any LOCC protocol Λ_{LOCC}.

Proof.
Let Λ_{LOCC} be an LOCC protocol leading to states σ_i^{AB} with probability p_i when applied to a state ρ^{AB}:

$$\Lambda_{\mathrm{LOCC}}[\rho^{AB}] = \sum_i p_i \sigma_i^{AB}.$$

Monotonicity of E_f under LOCC

Proposition 6.4. Entanglement of formation does not increase on average under local operations and classical communication:

$$\sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Theorem 6.1. Entanglement of formation does not increase under LOCC:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E_f(\rho)$$

for any LOCC protocol Λ_{LOCC}.

Proof.
We use Proposition 6.4. and convexity of E_f:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho^{AB}]) = E_f\left(\sum_i p_i \sigma_i^{AB}\right) \leq \sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Outline

Evaluating E_f for two qubits

- **Concurrence** of a two-qubit state ρ^{AB}:

$$C(\rho^{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Evaluating E_f for two qubits

- **Concurrence** of a two-qubit state ρ^{AB}:

$$C(\rho^{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

- λ_i: square roots (in decreasing order) of the eigenvalues of $\rho\tilde{\rho}$, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

  where ρ^* denotes entry-wise complex conjugation

Evaluating E_f for two qubits

- **Concurrence** of a two-qubit state ρ^{AB}:

$$C(\rho^{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

- λ_i: square roots (in decreasing order) of the eigenvalues of $\rho\tilde{\rho}$, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

  where ρ^* denotes entry-wise complex conjugation

- **Entanglement of formation:**

$$E_f(\rho^{AB}) = h\left(\frac{1 + \sqrt{1 - C^2(\rho^{AB})}}{2}\right)$$

  with $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$