Advanced quantum information: entanglement and nonlocality

Exam problems

Problem 1

For two quantum states ρ and σ the trace distance is given by

$$D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$$

where $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm of the matrix M. The von Neumann entropy of a quantum state ρ is defined as

$$S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i,$$

where λ_i are the eigenvalues of ρ. The quantum relative entropy between two states ρ and σ is given by

$$S(\rho\|\sigma) = \mathrm{Tr}[\rho \log_2 \rho] - \mathrm{Tr}[\rho \log_2 \sigma].$$

For a composite systems with two parties A and B of the same dimension $d = d_A = d_B$ the swap operation is given by

$$W^{AB} = \sum_{i,j=0}^{d-1} |i\rangle\langle j|^A \otimes |j\rangle\langle i|^B .$$

1. For two pure states (of arbitrary dimension) $|\psi\rangle$ and $|\phi\rangle$ calculate the trace distance $D(|\psi\rangle, |\phi\rangle)$ as a function of the overlap $\langle\psi|\phi\rangle$.
2. Show that the quantum relative entropy can take arbitrary large values, even if ρ and σ are qubit states.
3. For a bipartite state ρ^{AB}, express $S(\rho^{AB}\|\rho^A \otimes \rho^B)$ as a function of $S(\rho^A)$, $S(\rho^B)$, and $S(\rho^{AB})$, where $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$ and $\rho^B = \mathrm{Tr}_A[\rho^{AB}]$. What is the meaning of this quantity?
4. Show that the swap operation cannot create entanglement: for any separable state ρ^{AB}_{sep} also the state
$$\sigma^{AB} = W^{AB}\rho^{AB}_{\mathrm{sep}}\left(W^{AB}\right)^\dagger$$
is separable.
5. Show that the swap operation can create entanglement in a tripartite configuration. For this, give an example of a state ρ^{ABC} which has no entanglement between A and BC, but becomes entangled between A and BC after the swap operation acting on A and B.
6. Note that the maximally mixed qubit state can be written as
$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|),$$

but also as

$$\rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$$

with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. This means that the pure-state decomposition of the maximally mixed state is not unique. Can you find a physical interpretation for this property? Is it common for all quantum states? For a composite system, does there exist a separable state which admits a decomposition into maximally entangled states?

Problem 2

The tilted-CHSH functional is a generalisation of the CHSH functional which exhibits some qualitatively new features. In this problem your task will be to investigate some of them. The tilted-CHSH functional is defined as

$$\beta = \alpha\langle A_0\rangle + \langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle, \tag{1}$$

where $\alpha \in \mathbb{R}$ is a parameter.

1. Determine the local and no-signalling values of the tilted-CHSH functional for any $\alpha \in \mathbb{R}$.
2. For $\alpha \in [0, 2]$ the optimal quantum violation is achieved by Alice and Bob employing the following observables:

$$A_0 = \mathsf{Z}, \qquad B_0 = \cos b_\alpha \mathsf{Z} + \sin b_\alpha \mathsf{X}, \tag{2}$$

$$A_1 = \mathsf{X}, \qquad B_1 = \cos b_\alpha \mathsf{Z} - \sin b_\alpha \mathsf{X} \tag{3}$$

   for some angle $b_\alpha \in [0, \pi/2]$. Write down the resulting Bell operator in the Pauli basis.
3. Write down the same Bell operator in the computational basis (as a 4×4 matrix) and observe that it has a non-trivial block structure.
4. Given that

$$b_\alpha := \arcsin\left(\sqrt{\frac{4-\alpha^2}{8}}\right) \tag{4}$$

   compute the eigenvalues of the Bell operator. Remembering that this quantum realisation achieves the quantum value, plot the quantum value against the local and no-signalling values for $\alpha \in [0, 2]$.
5. Find the eigenstate corresponding to the largest eigenvalue of the Bell operator and write it down in its Schmidt form:

$$|\psi_\alpha\rangle := \cos\theta_\alpha |u_0\rangle|v_0\rangle + \sin\theta_\alpha |u_1\rangle|v_1\rangle\,. \tag{5}$$

   (Hint: to uniquely specify the Schmidt coefficients it is sufficient to give an expression for $\tan\theta_\alpha$)
6. Numerically compute the entanglement entropy of this state and plot it as a function of $\alpha \in [0, 2]$.

7. (More challenging) Having a complete description of the quantum realisation allows us to compute all the entries appearing in the NPA level 1 matrix (recall that in the observable-based picture this is a 5×5 matrix). Construct this matrix and compute its spectrum numerically for $\alpha = 0$, $\alpha = \frac{1}{2}$ and $\alpha = 1$.