

Advanced quantum information: entanglement and nonlocality

1. Homework sheet

Solutions to be submitted via email to a.streltsov@cent.uw.edu.pl

Please submit a single pdf file using "Solutions Advanced quantum information" in the subject line. Latest date for submission: 1. April 2020

Problem 1 (Lecture notes Section 1)

a) The density matrix ρ is positive semidefinite: $\langle \psi | \rho | \psi \rangle$ is a nonnegative real number

$$\langle \psi | \rho | \psi \rangle \geq 0$$

for any vector $|\psi\rangle$. Prove that any positive semidefinite matrix is Hermitian: $\rho = \rho^\dagger$.

b) A local quantum operation on Alice's side is defined as

$$\Lambda^A(\rho^{AB}) = \sum_i (K_i \otimes \mathbb{1}) \rho^{AB} (K_i \otimes \mathbb{1})^\dagger$$

with local Kraus operators K_i . Prove that the state of Bob does not change upon local operations on Alice's side.

c) A pure state $|\psi\rangle^{AB}$ is a purification of a mixed state ρ^A if

$$\rho^A = \text{Tr}_B [|\psi\rangle\langle\psi|^{AB}].$$

For $d_A = d_B$ prove that $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ are purifications of the same state if and only if

$$|\psi\rangle^{AB} = (\mathbb{1} \otimes U) |\phi\rangle^{AB}$$

for some local unitary U .

Problem 2 (Lecture notes Section 2)

a) Prove that a state $|\psi\rangle^{AB}$ is not entangled if and only if the reduced state ρ^A is pure.

b) Theorem 2.1. in the lecture notes states that a state $|\psi\rangle^{AB}$ can be converted into another state $|\phi\rangle^{AB}$ via LOCC if and only if $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$. Using this result, prove that the state

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

can be converted into any other state $|\psi\rangle^{AB}$ for $d_A = d_B = d$. Can Alice and Bob use LOCC to convert $|\Phi_d^+\rangle$ into a general mixed state?

c) For $d_A = d_B = 4$, assume that Alice and Bob share the state

$$|\psi\rangle^{AB} = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle.$$

Prove that by using LOCC this state cannot be converted into the state

$$|\phi\rangle^{AB} = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle.$$

Assume now that Alice and Bob can use a catalyst, an additional two-qubit pair in the state

$$|c\rangle^{A'B'} = \sqrt{0.6}|00\rangle + \sqrt{0.4}|11\rangle.$$

Prove that via LOCC Alice and Bob can convert $|\psi\rangle^{AB} \otimes |c\rangle^{A'B'}$ into $|\phi\rangle^{AB} \otimes |c\rangle^{A'B'}$.

d) For $d_A = d_B = d$ a state $|\Psi_d\rangle$ is called maximally entangled if

$$\text{Tr}_A[|\Psi_d\rangle\langle\Psi_d|] = \frac{1}{d}\mathbb{1}_d,$$

where $\mathbb{1}_d$ denotes the $d \times d$ identity matrix. Prove that for any maximally entangled state $|\Psi_d\rangle$ there exist local unitaries U and V such that

$$|\Psi_d\rangle = (U \otimes \mathbb{1})|\Phi_d^+\rangle = (\mathbb{1} \otimes V)|\Phi_d^+\rangle.$$

Problem 3 (Lecture notes Section 3)

a) For a bipartite state ρ^{AB} , the partial transpose with respect to Alice is denoted ρ^{T_A} , while ρ^{T_B} denotes partial transpose with respect to Bob. Prove that ρ^{T_A} and ρ^{T_B} have the same eigenvalues.

b) Consider two matrices M^{AB} and N^{AB} acting on the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. M^{T_B} denotes the partial transpose of M^{AB} , and N^{T_B} denotes the partial transpose of N^{AB} . Prove that

$$\text{Tr}[M^{T_B}N] = \text{Tr}[MN^{T_B}].$$

Hint: note that any matrix M^{AB} can be expanded as $M^{AB} = \sum_{i,j} |i\rangle\langle j| \otimes M_{ij}^B$ with some matrices M_{ij}^B acting on \mathcal{H}_B .

c) For $d_A = d_B = 2$ consider the state

$$\rho^{AB} = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{1}_4}{4}, \quad (1)$$

where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state and p is ranging between 0 and 1. For which values of p is the state entangled? The state (1) is also called Werner state.

d) For a bipartite state ρ^{AB} define

$$\sigma^{AB} = (U \otimes V)\rho^{AB}(U \otimes V)^\dagger$$

with local unitaries U and V . Prove that ρ^{T_A} and σ^{T_A} have the same eigenvalues.