Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

3rd class
March 18, 2020

Advanced quantum information

- Every Wednesday 14:15 - 16:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al*., *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- Homework to be submitted via email as a single pdf
- **If you have a question during the class:**
  - **unmute audio first**
  - **mute audio when finished**
- **To allow to adjust your devices,**
  today's class will start at 14:20
- **If you don't hear music now: press "join audio"**

Outline

1. Theory of quantum entanglement
2. Entanglement detection
3. Shannon entropy, von Neumann entropy, and typical sequences

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1. Theory of quantum entanglement

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Reminder: Majorization

Consider two real d-dimensional vectors $\vec{x}$ and $\vec{y}$ with elements in decreasing order. Then $\vec{x} \prec \vec{y}$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

for all $k \in [1, d-1]$, and $\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$.

Pure state conversion

Theorem 2.1. There exists an LOCC protocol transforming $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ if and only if $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$, where $\vec{\lambda}_\psi$ denotes the vector with eigenvalues of the reduced state $\mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$ in decreasing order.

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Catalytic conversion: if there is no LOCC protocol such that $|\psi\rangle^{AB} \to |\phi\rangle^{AB}$, there might be a catalyst state $|c\rangle$ such that

$$|\psi\rangle^{AB} \otimes |c\rangle^{A'B'} \to |\phi\rangle^{AB} \otimes |c\rangle^{A'B'}$$

Bell states

Bell states (or EPR states):

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Phi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

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- Reduced state of any Bell state: $\frac{1}{2}\mathbb{1}_2$
- For any single-qubit state ρ it holds $\frac{1}{2}\mathbb{1}_2 \prec \rho$
- Theorem 2.1. $\Rightarrow$ **any Bell state can be converted into any two-qubit pure state via LOCC**

Maximally entangled states

Bell states are also called **maximally entangled states**

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- For $d_A = d_B = d$ a state $|\Psi_d\rangle$ is maximally entangled if and only if

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- All maximally entangled states are equivalent to

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$$

up to local unitary on one side:

$$|\Psi_d\rangle = \big(U\otimes\mathbb{1}\big)\,|\Phi_d^+\rangle = \big(\mathbb{1}\otimes V\big)\,|\Phi_d^+\rangle$$

Entanglement for mixed states

Separable mixed states:

$$\rho_{\text{sep}}^{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

with $p_i \geq 0$, $\sum_i p_i = 1$, $|\psi_i\rangle \in \mathcal{H}_A$ and $|\phi_i\rangle \in \mathcal{H}_B$.

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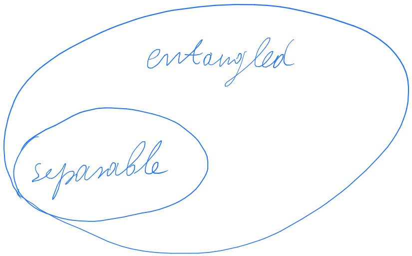
States which are not separable are called **entangled**



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Entanglement witnesses

Entanglement witness: Hermitian matrix W^{AB} such that

$$\mathrm{Tr}\left[W^{AB}\left(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|\right)\right]=\left(\langle\psi|\otimes\langle\phi|\right)W^{AB}\left(|\psi\rangle\otimes|\phi\rangle\right)\geq 0$$

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- For any separable state ρ^{AB}_{sep} we have

$$\begin{aligned}\mathrm{Tr}\left[W^{AB}\rho^{AB}_{\mathrm{sep}}\right]&=\mathrm{Tr}\left[W^{AB}\left(\sum_i p_i\,|\psi_i\rangle\langle\psi_i|\otimes|\phi_i\rangle\langle\phi_i|\right)\right]\\&=\sum_i p_i\,\mathrm{Tr}\left[W^{AB}\left(|\psi_i\rangle\langle\psi_i|\otimes|\phi_i\rangle\langle\phi_i|\right)\right]\geq 0\end{aligned}$$

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- If $\mathrm{Tr}\left[W^{AB}\rho^{AB}\right]<0$, the state ρ^{AB} must be entangled

Entanglement witnesses

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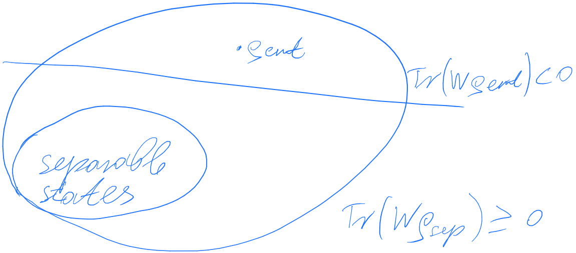

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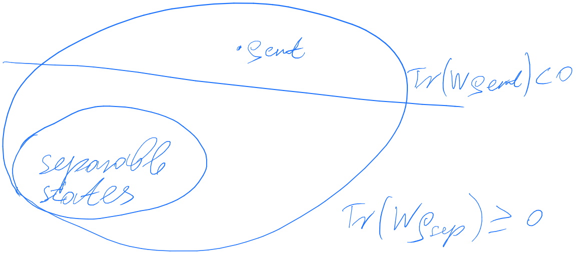

Interpretation of W^{AB}: observable with expectation value $\mathrm{Tr}\left[W^{AB}\rho^{AB}\right]$

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Example. Swap operation for $d_A = d_B$:

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$$\left(\langle\psi| \otimes \langle\phi|\right) W^{AB} \left(|\psi\rangle \otimes |\phi\rangle\right) = \left(\langle\psi| \otimes \langle\phi|\right) \left(|\phi\rangle \otimes |\psi\rangle\right) = |\langle\psi|\phi\rangle|^2 \geq 0$$

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$$W^{AB} |\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(W^{AB}|01\rangle - W^{AB}|10\rangle\Big) = -|\Psi^-\rangle$$

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- $\Rightarrow W^{AB}$ detects entanglement in $|\Psi^-\rangle$

Partial transposition

Partial transposition on Bob's subsystem:

$$\rho^{T_B} = \left(\sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \right)^{T_B}$$
$$= \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes \big(|k\rangle\langle l|\big)^{T}$$
$$= \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |l\rangle\langle k|$$

Partial transposition

Example. For $|\psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$ we have

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

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$$\rho^{T_A} = \begin{pmatrix} X^T & Y^T \\ \left(Y^\dagger\right)^T & Z^T \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & 0 \\ 0 & 0 & \cos\alpha\sin\alpha & 0 \\ 0 & \cos\alpha\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & \sin^2\alpha \end{pmatrix}$$

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Eigenvalues of ρ^{T_A}: $\cos^2\alpha$, $\sin^2\alpha$, $\pm|\cos\alpha\sin\alpha|$

$\Rightarrow |\psi\rangle$ is entangled for all $\alpha \neq n\frac{\pi}{2}$

Partial transposition

Applying partial transposition to a separable state:

$$\rho_{\text{sep}}^{T_B} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes \big(|\phi_i\rangle\langle\phi_i|\big)^T = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i^*\rangle\langle\phi_i^*|$$

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$\Rightarrow$ **PPT criterion:** if ρ^{T_B} is not positive, ρ must be entangled

Positive and completely positive maps

Positive map: linear map Λ acting on matrices such that $\Lambda(\rho)$ is positive semidefinite for any positive semidefinite matrix ρ

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For a bipartite density matrix ρ^{AB} we define

$$\mathbb{1} \otimes \Lambda(\rho^{AB}) = \mathbb{1} \otimes \Lambda\left(\sum_{i,j,k,l} c_{ijkl}\, |i\rangle\langle j| \otimes |k\rangle\langle l|\right) = \sum_{i,j,k,l} c_{ijkl}\, |i\rangle\langle j| \otimes \Lambda(|k\rangle\langle l|)$$

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Completely positive (CP) map: a positive map Λ such that $\mathbb{1} \otimes \Lambda(\rho^{AB})$ is positive for any positive semidefinite matrix ρ^{AB} in the extended Hilbert space of any dimension

Choi–Jamiołkowski isomorphism

Choi matrix of a linear map Λ:

$$M_\Lambda = (\mathbb{1} \otimes \Lambda) |\Phi_d^+\rangle\langle\Phi_d^+| = \frac{1}{d} \sum_{i,j} |i\rangle\langle j| \otimes \Lambda\left(|i\rangle\langle j|\right)$$

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Choi-Jamiołkowski isomorphism:

- Λ is a positive map if and only if M_Λ is an entanglement witness
- For any entanglement witness W^{AB} there exists a positive map Λ such that $W^{AB} = M_\Lambda$
- Λ is completely positive if and only if M_Λ is positive semidefinite

PPT criterion for two qubits

Proposition 3.1. For $d_A = d_B = 2$ a state ρ^{AB} is separable if and only if ρ^{T_B} is positive semidefinite.

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Proof. For any entangled state ρ^{AB} there exists an entanglement witness W^{AB} such that (see Theorem 3.1.)

$$\mathrm{Tr}\left[W^{AB}\rho^{AB}\right] < 0.$$

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$$\mathrm{Tr}\left[W^{AB}\rho^{AB}\right] < 0.$$

With the Choi-Jamiołkowski isomorphism, there also exists a positive map Λ such that

$$\mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda\, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] < 0.$$

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Every positive qubit map can be decomposed as

$$\Lambda(\rho) = \Lambda_1^{\mathrm{CP}}(\rho) + \left[\Lambda_2^{\mathrm{CP}}(\rho)\right]^T$$

with CP maps Λ_i^{CP}.

PPT criterion for two qubits

There exists a positive map Λ such that

$$\mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda\, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] < 0.$$

Thus,

$$\begin{aligned} 0 > \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda\, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] &= \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda_1^{\mathrm{CP}}\, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] \\ &+ \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda_2^{\mathrm{CP}}\, |\Phi^+\rangle\langle\Phi^+|\right)^{T_B}\rho^{AB}\right] \\ &= \mathrm{Tr}\left[X_1\rho^{AB}\right] + \mathrm{Tr}\left[X_2^{T_B}\rho^{AB}\right] \end{aligned}$$

with positive matrices $X_i = \mathbb{1} \otimes \Lambda_i^{\mathrm{CP}}\, |\Phi^+\rangle\langle\Phi^+|$.

PPT criterion for two qubits

In summary,

$$0 > \mathrm{Tr}\left[X_1\rho^{AB}\right] + \mathrm{Tr}\left[X_2^{T_B}\rho^{AB}\right]$$

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In summary,

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with positive matrices $X_i = \mathbb{1} \otimes \Lambda_i^{\mathrm{CP}} |\Phi^+\rangle\langle\Phi^+|$.

Using

$$\mathrm{Tr}\left[X_2^{T_B} \rho^{AB}\right] = \mathrm{Tr}\left[X_2 \rho^{T_B}\right]$$

we obtain

$$0 > \mathrm{Tr}\left[X_1 \rho^{AB}\right] + \mathrm{Tr}\left[X_2 \rho^{T_B}\right] \geq \mathrm{Tr}\left[X_2 \rho^{T_B}\right]$$

Since X_2 is positive, ρ^{T_B} must have negative eigenvalues.
Q.E.D.

PPT criterion for two qubits

For larger dimensions:

PPT criterion for two qubits

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Theorem 3.2. For $d_A d_B \leq 6$ a state ρ^{AB} is separable if and only if ρ^{T_B} is positive. For all $d_A d_B > 6$ there exist entangled states which have positive partial transpose.

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3. Shannon entropy, von Neumann entropy, and typical sequences

Shannon and von Neumann entropy

Consider an integer random variable x with probability distribution $p(x)$. A sequence of **independent and identically distributed** variables x_i has probability distribution

$$p(x_1, \ldots, x_m) = p(x_1)p(x_2)\ldots p(x_m).$$

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Von Neumann entropy of a quantum state ρ with eigenvalues λ_i:

$$S(\rho) = -\operatorname{Tr}[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i$$

Typical sequences

ϵ-typical sequence: sequence of independent and identically distributed random variables x_i such that

$$2^{-m(H(p(x))+\epsilon)} \leq p(x_1, \ldots, x_m) \leq 2^{-m(H(p(x))-\epsilon)}$$

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Theorem of typical sequences:
(1) Fix $\epsilon > 0$. For any $\delta > 0$, for sufficiently large m the probability that a sequence is ϵ-typical is at least $1 - \delta$.

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Theorem of typical sequences:
(1) Fix $\epsilon > 0$. For any $\delta > 0$, for sufficiently large m the probability that a sequence is ϵ-typical is at least $1 - \delta$.
(2) For any fixed $\epsilon > 0$ and $\delta > 0$, for sufficiently large m, the number $|T(m, \epsilon)|$ of ϵ-typical sequences satisfies

$$(1 - \delta)2^{m(H(p(x))-\epsilon)} \leq |T(m, \epsilon)| \leq 2^{m(H(p(x))+\epsilon)}.$$