Advanced quantum information: entanglement and nonlocality

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5th class
April 1st, 2020

Advanced quantum information (5th class)

- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/

- These slides available online now

- If you have a question during the class:
  - unmute audio first
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- **Next class: April 15 (in 2 weeks)**

Outline

1. Entanglement distillation and dilution
   - Reminder: pure state entanglement distillation and dilution
   - Mixed state entanglement distillation
   - Matrix realignment criterion
   - Bound entanglement
2. Entanglement quantification

Outline

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Reminder: pure state entanglement distillation and dilution

Proposition 5.1. The entanglement cost of a state $|\psi\rangle$ is at most $S(\rho_\psi)$.

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Reminder: pure state entanglement distillation and dilution

Proposition 5.1. The entanglement cost of a state $|\psi\rangle$ is at most $S(\rho_\psi)$.

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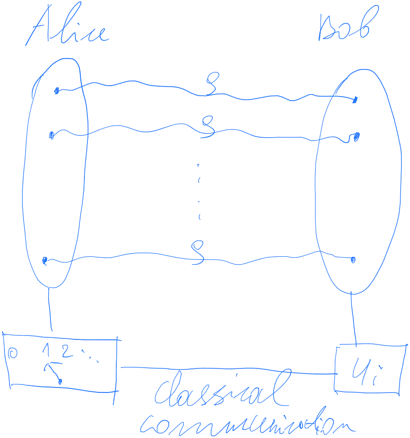
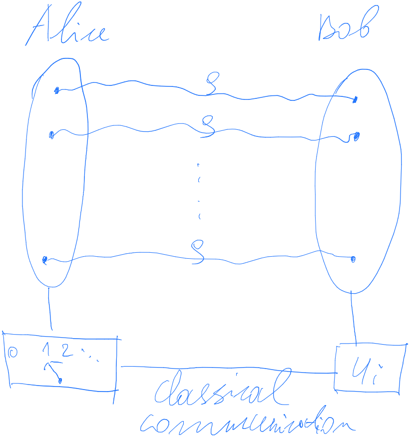
Theorem 5.1. The distillable entanglement and entanglement cost of a state $|\psi\rangle$ are equal to $S(\rho_\psi)$.

Outline

Mixed state entanglement distillation

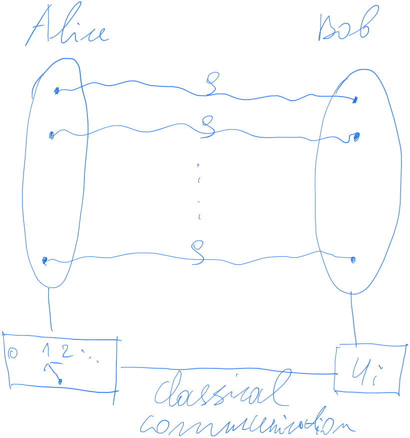


Mixed state entanglement distillation



Entanglement distillation for **mixed states:** converting m copies of ρ into n singlets in the limit $m \to \infty$

Mixed state entanglement distillation



Entanglement distillation for **mixed states:** converting m copies of ρ into n singlets in the limit $m \to \infty$

Exercise: can a separable state $\rho_{\text{sep}} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$ be distilled into singlets?

Mixed state entanglement distillation

Separable states cannot be distilled into singlets:

Mixed state entanglement distillation

Separable states cannot be distilled into singlets:

- Section 5.5.: stochastic LOCC brings $\rho^{\otimes m}$ to

$$\sigma = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

with probability $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$

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  with probability $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$

- If ρ is separable $\Rightarrow \rho^{\otimes m}$ is separable $\Rightarrow \sigma$ is separable

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

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Proof.

- Assume ρ can be distilled into singlets

Mixed state entanglement distillation

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- Assume ρ can be distilled into singlets
- $\Rightarrow$ There exists a stochastic LOCC protocol bringing $\rho^{\otimes m}$ arbitrary close to a singlet for large m

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.

- Assume ρ can be distilled into singlets
- $\Rightarrow$ There exists a stochastic LOCC protocol bringing $\rho^{\otimes m}$ arbitrary close to a singlet for large m
- There must exist a stochastic LOCC protocol transforming $\rho^{\otimes m}$ into an entangled two-qubit state σ_{2q}

Mixed state entanglement distillation

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Proof.

- Section 5.5.: stochastic LOCC transformation has the form

$$\sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$$

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- Probability: $p = \mathrm{Tr}[\sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger]$
- A_j and B_j: $2 \times d_A$ and $2 \times d_B$ rectangular matrices

Mixed state entanglement distillation

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Proof.

- ρ distillable $\Rightarrow \sigma_{2q} = \frac{1}{p} \sum_j A_j \otimes B_j \rho^{\otimes m} A_j^\dagger \otimes B_j^\dagger$ is entangled

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- $\Rightarrow \sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i

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- Probability: $p_i = \mathrm{Tr}[A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger]$
- **Exercise:** prove that for entangled state σ_{2q} there must exist i such that σ_i is entangled

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- Probability: $p_i = \mathrm{Tr}[A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger]$
- **Exercise:** prove that for entangled state σ_{2q} there must exist i such that σ_i is entangled
- **Solution:** note that $\sigma_{2q} = \frac{1}{\sum_j p_j} \sum_i p_i \sigma_i$
  $\Rightarrow \sigma_{2q}$ is separable if all σ_i are separable

Mixed state entanglement distillation

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- $\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- A_i and B_i: rectangular $2 \times d_A$ and $2 \times d_B$ matrices

Mixed state entanglement distillation

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- $\sigma_i = \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger$ is entangled for some i
- A_i and B_i: rectangular $2 \times d_A$ and $2 \times d_B$ matrices
- It follows:

$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
$$B_i = |0\rangle\langle\beta_0| + |1\rangle\langle\beta_1|$$

Mixed state entanglement distillation

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- $|\alpha_i\rangle \in \mathcal{H}_A$ and $|\beta_i\rangle \in \mathcal{H}_B$ are (possibly unnormalized) vectors

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$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
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- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$

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$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
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- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$
- P_B: projector onto the subspace spanned by $|\beta_0\rangle$ and $|\beta_1\rangle$

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$$A_i = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|$$
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- P_A: projector onto the subspace spanned by $|\alpha_0\rangle$ and $|\alpha_1\rangle$
- P_B: projector onto the subspace spanned by $|\beta_0\rangle$ and $|\beta_1\rangle$
- It holds

$$\begin{aligned}\sigma_i &= \frac{1}{p_i} A_i \otimes B_i \rho^{\otimes m} A_i^\dagger \otimes B_i^\dagger \\ &= \frac{1}{p_i} A_i \otimes B_i \left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B \right) A_i^\dagger \otimes B_i^\dagger\end{aligned}$$

Mixed state entanglement distillation

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Proof.

- It holds

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- σ_i is entangled $\Rightarrow$

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B \right]}$$

  must be entangled

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- Consider orthonormal product basis $|f_i\rangle \otimes |g_k\rangle$ such that

$$\begin{aligned} P_A &= |f_0\rangle\langle f_0| + |f_1\rangle\langle f_1| \\ P_B &= |g_0\rangle\langle g_0| + |g_1\rangle\langle g_1| \end{aligned}$$

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- In the basis $|f_i\rangle \otimes |g_k\rangle$ the state μ takes the form

$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\operatorname{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

Mixed state entanglement distillation

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- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$

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- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$
- If $\tau_{2q}^{T_A}$ is positive $\Rightarrow$ τ_{2q} is separable (Theorem 3.2.)

Mixed state entanglement distillation

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- For evaluating μ^{T_A} we can focus on $\tau_{2q}^{T_A}$
- If $\tau_{2q}^{T_A}$ is positive $\Rightarrow \tau_{2q}$ is separable (Theorem 3.2.)
- $\Rightarrow$ **Contradiction:** μ is separable but we showed previously that μ must be entangled!
- $\Rightarrow \tau_{2q}^{T_A}$ must have negative eigenvalues

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$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

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$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

There exists a vector

$$|\psi\rangle = \sum_{i,k=0}^{1} c_{ik} |f_i\rangle |g_k\rangle$$

such that $\langle\psi|\tau_{2q}^{T_A}|\psi\rangle < 0$

Mixed state entanglement distillation

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$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \begin{pmatrix} \tau_{2q} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

and $\tau_{2q}^{T_A}$ must have negative eigenvalues

We have $\langle\psi|\tau_{2q}^{T_A}|\psi\rangle = \langle\psi|\mu^{T_A}|\psi\rangle$, which implies that

$$\langle\psi|\mu^{T_A}|\psi\rangle < 0$$

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$$\mu = \frac{P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}, \qquad \langle\psi|\mu^{T_A}|\psi\rangle < 0$$

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The following equalities hold:

$$\left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right)^{T_A} = P_A \otimes P_B \left(\rho^{\otimes m}\right)^{T_A} P_A \otimes P_B,$$
$$P_A \otimes P_B \,|\psi\rangle = |\psi\rangle$$

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$$P_A \otimes P_B \left|\psi\right\rangle = \left|\psi\right\rangle$$

We obtain:

$$0 > \langle\psi|\mu^{T_A}|\psi\rangle = \frac{\langle\psi| \left(P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right)^{T_A} |\psi\rangle}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} =$$
$$= \frac{\langle\psi| P_A \otimes P_B \left(\rho^{\otimes m}\right)^{T_A} P_A \otimes P_B |\psi\rangle}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]} = \frac{\langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle}{\mathrm{Tr}\left[P_A \otimes P_B \rho^{\otimes m} P_A \otimes P_B\right]}$$

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

Mixed state entanglement distillation

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Exercise: prove that ρ^{T_A} is not positive semidefinite

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Solution:

- for two matrices $M_1^{A_1 B_1}$ and $M_2^{A_2 B_2}$ it holds that

$$\left(M_1^{A_1 B_1} \otimes M_2^{A_2 B_2}\right)^{T_{A_1 A_2}} = \left(M_1^{A_1 B_1}\right)^{T_{A_1}} \otimes \left(M_2^{A_2 B_2}\right)^{T_{A_2}},$$

  and similar for more than 2 matrices

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  and similar for more than 2 matrices
- $\Rightarrow \left(\rho^{\otimes m}\right)^{T_A} = \left(\rho^{T_A}\right)^{\otimes m}$

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Proof.
We have $0 > \langle\psi| \left(\rho^{\otimes m}\right)^{T_A} |\psi\rangle$

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Solution:

- for two matrices $M_1^{A_1 B_1}$ and $M_2^{A_2 B_2}$ it holds that

$$\left(M_1^{A_1 B_1} \otimes M_2^{A_2 B_2}\right)^{T_{A_1 A_2}} = \left(M_1^{A_1 B_1}\right)^{T_{A_1}} \otimes \left(M_2^{A_2 B_2}\right)^{T_{A_2}},$$

  and similar for more than 2 matrices
- $\Rightarrow \left(\rho^{\otimes m}\right)^{T_A} = \left(\rho^{T_A}\right)^{\otimes m}$
- $\Rightarrow \rho^{T_A}$ must have negative eigenvalues
  Q.E.D.

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

Mixed state entanglement distillation

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- Separable states have positive partial transpose

Mixed state entanglement distillation

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- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled
- **Are there entangled states which cannot be distilled?**

Mixed state entanglement distillation

Theorem 5.2. States with positive partial transpose cannot be distilled into singlets.

- Separable states have positive partial transpose
- $\Rightarrow$ Separable states cannot be distilled
- **Are there entangled states which cannot be distilled?**
- Independent entanglement detection criterion required

Outline

Matrix realignment criterion

- Consider a 2×2 matrix

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

Matrix realignment criterion

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$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

- We can “vectorize” M by defining

$$\vec{M} = (M_{00}, M_{10}, M_{01}, M_{11})^T$$

Matrix realignment criterion

- Consider a 2×2 matrix

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$$

- We can "vectorize" M by defining

$$\vec{M} = (M_{00}, M_{10}, M_{01}, M_{11})^T$$

- Similar for larger dimensions

Matrix realignment criterion

- Consider two-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^{\dagger} & Z \end{pmatrix}$$

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$$\|M\|_1 = \text{Tr}\,\sqrt{M^\dagger M} = \sum_i s_i$$
- Triangle inequality:
$$\|A + B\|_1 \le \|A\|_1 + \|B\|_1$$
- Trace norm is absolutely homogeneous:
$$\|aM\|_1 = |a| \cdot \|M\|_1$$
for any matrix M and any $a \in \mathbb{C}$

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Proposition 5.3. Any separable state ρ fulfills $||\tilde{\rho}||_1 \leq 1$.

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Trace norm of $\widetilde{\rho_{\text{sep}}}$:

$$\|\widetilde{\rho_{\text{sep}}}\|_1 = \left\| \sum_i p_i \overrightarrow{\psi}_i \cdot \overrightarrow{\phi}_i^T \right\|_1 \leq \sum_i p_i \left\| \overrightarrow{\psi}_i \cdot \overrightarrow{\phi}_i^T \right\|_1 = 1$$

Q.E.D.

Outline

Bound entanglement

For $d_A = d_B = 3$ and $0 \leq a \leq 1$ consider

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}$$

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- ρ_a is PPT for all $0 \leq a \leq 1$
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- $\Rightarrow \rho_a$ is bound entangled for $0 < a < 1$

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- All separable states and some entangled states have positive partial transpose (PPT)
- PPT states cannot be distilled into singlets
- There are PPT entangled states $\Rightarrow$ these states require singlets to be created, but cannot be converted into singlets
- These states are called **bound entangled**

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How much entanglement is in a given quantum state ρ?

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Theorem 2.1. $\Rightarrow |\Psi_d\rangle$ has maximum entanglement

Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

where $\rho^A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

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- **Interpretation:** minimal average entanglement required to create ρ^{AB}

Entanglement of formation

Proposition 6.1. For pure states $|\psi\rangle^{AB}$ entanglement of formation does not increase on average under local measurements on Alices side:

$$\sum_i p_i E_f(|\phi_i\rangle^{AB}) \leq E_f(|\psi\rangle^{AB})$$

with

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1}\,|\psi\rangle\langle\psi|^{AB}\,K_i^\dagger \otimes \mathbb{1}\right],$$
$$|\phi_i\rangle^{AB} = \frac{1}{\sqrt{p_i}}(K_i \otimes \mathbb{1})\,|\psi\rangle^{AB}\,.$$

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Proof.

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- By definition of E_f we have

$$E_f(|\psi\rangle^{AB}) = S(\rho^B), \qquad \sum_i p_i E_f(|\phi_i\rangle^{AB}) = \sum_i p_i S(\sigma_i^B).$$

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$$\begin{aligned} \sum_i p_i E_f(|\phi_i\rangle^{AB}) &= \sum_i p_i S(\sigma_i^B) \leq S\left(\sum_i p_i \sigma_i^B\right) \\ &= S(\rho^B) = E_f(|\psi\rangle^{AB}) \end{aligned}$$

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- In summary: $\sum_i p_i E_f(|\phi_i\rangle^{AB}) \leq E_f(|\psi\rangle^{AB})$
  Q.E.D.