Advanced quantum information: entanglement and nonlocality

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6th class
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Advanced quantum information (6th class)

- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- Comments on 1. homework sheet have been sent on 7. April
- These slides available online now
- If you have a question during the class:
  - unmute audio first
  - mute audio when finished
- If you don’t hear music now: press “join audio”

Outline

1. General entanglement measures
2. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits
3. Negativity E_n

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Theorem 2.1.: $|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ can be converted into any other state ρ via LOCC $\Rightarrow$ $|\Phi_d^+\rangle$ has maximum entanglement

$$E(\rho) = E\left(\Lambda_{\mathrm{LOCC}}\left[|\Phi_d^+\rangle\langle\Phi_d^+|\right]\right) \leq E(|\Phi_d^+\rangle)$$

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Entanglement of formation

- Entanglement of formation for pure states:

$$E_f(|\psi\rangle^{AB}) = S(\rho^A),$$

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- **Interpretation:** minimal average entanglement required to create ρ^{AB}

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- For any decomposition $\{p_i, |\psi_i\rangle^{AB}\}$ the average entanglement $\sum_i p_i E_f(|\psi_i\rangle^{AB})$ is nonnegative

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Solution:

- For any decomposition $\{p_i, |\psi_i\rangle^{AB}\}$ the average entanglement $\sum_i p_i E_f(|\psi_i\rangle^{AB})$ is nonnegative
- For a separable state σ^{AB} there exists a decompotision into product states $|\psi_i\rangle^{AB} = |\alpha_i\rangle^A \otimes |\beta_i\rangle^B$ with $E_f(|\psi_i\rangle^{AB}) = 0$

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$\Rightarrow$ in combination, this will prove that E_f does not increase under LOCC

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Proposition 6.1. Entanglement of formation is convex:

$$E_f\left(\sum_i p_i\rho_i^{AB}\right) \le \sum_i p_i E_f\left(\rho_i^{AB}\right).$$

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Proof.

- Consider a decomposition of $\rho_i^{AB} = \sum_j q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$ with the property that

$$E_f(\rho_i^{AB}) = \sum_j q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right).$$

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- Defining $\sigma^{AB} = \sum_i p_i \rho_i^{AB}$ we obtain

$$\sigma^{AB} = \sum_i p_i \rho_i^{AB} = \sum_{ij} p_i q_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|^{AB}$$

$$\sum_i p_i E_f\left(\rho_i^{AB}\right) = \sum_{ij} p_i q_{ij} E_f\left(|\psi_{ij}\rangle^{AB}\right)$$

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- E_f is the minimal average entanglement $\Rightarrow$

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- In summary: $E_f\left(\sum_i p_i \rho_i^{AB}\right) = E_f\left(\sigma^{AB}\right) \leq \sum_i p_i E_f\left(\rho_i^{AB}\right)$. Q.E.D.

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Monotonicity of E_f under local measurements

Proposition 6.2. For pure states $|\psi\rangle^{AB}$ entanglement of formation does not increase on average under local measurements on Alice's side:

$$\sum_i p_i E_f(|\phi_i\rangle^{AB}) \leq E_f(|\psi\rangle^{AB})$$

with

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1}\, |\psi\rangle\langle\psi|^{AB}\, K_i^\dagger \otimes \mathbb{1}\right],$$
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Proof. See last class.

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Extension to mixed states ρ^{AB} and local Kraus operators K_i:

$$p_i = \mathrm{Tr}\left[K_i \otimes \mathbb{1}\rho^{AB} K_i^\dagger \otimes \mathbb{1}\right]$$
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- Note that $\sum_j q_j p_{ij} = p_i$

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For the entanglement of formation of σ_i^{AB} we obtain

$$E_f\left(\sigma_i^{AB}\right) = E_f\left(\frac{1}{p_i} K_i \otimes \mathbb{1} \rho^{AB} K_i^\dagger \otimes \mathbb{1}\right)$$

$$= E_f\left(\sum_j \frac{q_j}{p_i} K_i \otimes \mathbb{1} \, |\psi_j\rangle\langle\psi_j|^{AB} \, K_i^\dagger \otimes \mathbb{1}\right)$$

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- $\sum_i p_i E_f\left(\sigma_i^{AB}\right) \leq \sum_j q_j \sum_i p_{ij} E_f\left(|\phi_{ij}\rangle^{AB}\right) \leq \sum_j q_j E_f\left(|\psi_j\rangle^{AB}\right)$

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- Recall that $\{q_j, |\psi_j\rangle^{AB}\}$ is an optimal decomposition:

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Generalizes to local measurements on Alice's and Bob's side, with exchange of measurement outcomes via classical channel

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Exercise: prove this theorem from Proposition 6.4. by using convexity of E_f

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Proof.
Let Λ_{LOCC} be an LOCC protocol leading to states σ_i^{AB} with probability p_i when applied to a state ρ^{AB}:

$$\Lambda_{\mathrm{LOCC}}[\rho^{AB}] = \sum_i p_i \sigma_i^{AB}.$$

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Proof.
We use Proposition 6.4. and convexity of E_f:

$$E_f(\Lambda_{\mathrm{LOCC}}[\rho^{AB}]) = E_f\left(\sum_i p_i \sigma_i^{AB}\right) \leq \sum_i p_i E_f(\sigma_i^{AB}) \leq E_f(\rho^{AB}).$$

Outline

1. General entanglement measures
2. Entanglement of formation E_f
   - Convexity of E_f
   - Monotonicity of E_f under local measurements
   - Monotonicity of E_f under LOCC
   - Evaluating E_f for two qubits
3. Negativity E_n

Evaluating E_f for two qubits

- **Concurrence** of a two-qubit state ρ^{AB}:

$$C(\rho^{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

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- **Entanglement of formation:**

$$E_f(\rho^{AB}) = h\left(\frac{1 + \sqrt{1 - C^2(\rho^{AB})}}{2}\right)$$

  with $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$

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- For any Hermitian matrix H the trace norm is monotonic under quantum operations:

$$\left\|\Lambda[H]\right\|_1 \leq \|H\|_1$$

  for any quantum operation Λ.

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Theorem 6.2. Negativity does not increase under LOCC:

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$\Rightarrow A_i \otimes B_i^*$ are also valid Kraus operators

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Proof.

- Partial transpose of $\Lambda_{\mathrm{LOCC}}[\rho^{AB}]$:

$$\begin{aligned}\left(\Lambda_{\mathrm{LOCC}}\left[\rho^{AB}\right]\right)^{T_B} &= \left(\sum_i A_i \otimes B_i \rho^{AB} A_i^\dagger \otimes B_i^\dagger\right)^{T_B} \\ &= \sum_i A_i \otimes B_i^* \rho^{T_B} A_i^\dagger \otimes B_i^T\end{aligned}$$

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- Taking the trace norm gives

$$\left\|\left(\Lambda_{\mathrm{LOCC}}\left[\rho^{AB}\right]\right)^{T_B}\right\|_1 = \left\|\sum_i A_i \otimes B_i^* \rho^{T_B} A_i^\dagger \otimes B_i^T\right\|_1 = \left\|\tilde{\Lambda}\left[\rho^{T_B}\right]\right\|_1$$

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- In summary:

$$\left\|\left(\Lambda_{\mathrm{LOCC}}\left[\rho^{AB}\right]\right)^{T_B}\right\|_1 \leq \left\|\rho^{T_B}\right\|_1$$

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Recall definition of negativity:

$$E_n(\rho^{AB}) = \frac{\|\rho^{T_B}\|_1 - 1}{2}$$

Q.E.D.