Advanced quantum information: entanglement and nonlocality

1. Homework sheet

Solutions to be submitted via email to m.scalici@cent.uw.edu.pl
Please submit a single pdf file using "Solutions Advanced Quantum Information" in the subject line. Latest date for submission: 22. March 2022

Problem 1 (Lecture notes Section 1)

a) The density matrix ρ is positive semidefinite: $\langle\psi|\rho|\psi\rangle$ is a nonnegative real number

$$\langle\psi|\rho|\psi\rangle \geq 0$$

for any vector $|\psi\rangle$. Prove that any positive semidefinite matrix is Hermitian: $\rho = \rho^\dagger$.

b) A local quantum operation on Alice's side is defined as

$$\Lambda^A(\rho^{AB}) = \sum_i (K_i \otimes \mathbb{1})\rho^{AB}(K_i \otimes \mathbb{1})^\dagger$$

with local Kraus operators K_i. Prove that the state of Bob does not change upon local operations on Alice's side.

c) A pure state $|\psi\rangle^{AB}$ is a purification of a mixed state ρ^A if

$$\rho^A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|^{AB}\right].$$

For $d_A = d_B$ prove that $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ are purifications of the same state if and only if

$$|\psi\rangle^{AB} = (\mathbb{1} \otimes U)\,|\phi\rangle^{AB}$$

for some local unitary U.

d)[1] Prove that for every local measurement with Kraus operators $\left\{K_i^A\right\}_{i=1}^n$ there exist a unitary $U = U^{AB}$ and a complete set of projectors $\{\Pi_i\}_{i=1}^n$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\mathrm{Tr}_B\left[\Pi_i U\left(\rho^A \otimes |0\rangle\langle 0|^B\right)U^\dagger \Pi_i\right] = K_i^A \rho^A \left(K_i^A\right)^\dagger$$

for all $1 \leq i \leq n$ and every density matrix ρ^A acting on $\mathcal{H}_A$. Using this result, prove that for every local POVM $\left\{M_i^A\right\}_{i=1}^n$ there exists a complete set of projectors $\{\Pi_i\}_{i=1}^n$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\mathrm{Tr}\left[M_i^A \rho^A\right] = \mathrm{Tr}\left[\Pi_i \rho^A \otimes |0\rangle\langle 0|^B\right]$$

for all $1 \leq i \leq n$ and every density matrix ρ^A acting on $\mathcal{H}_A$.

[1] Problem 1d) is optional and does not contribute to the total number of points of the homework sheet.

Problem 2 (Lecture notes Section 2)

a) Prove that a state $|\psi\rangle^{AB}$ is not entangled if and only if the reduced state ρ^A is pure.

b) Theorem 2.1. in the lecture notes states that a state $|\psi\rangle^{AB}$ can be converted into another state $|\phi\rangle^{AB}$ via LOCC if and only if $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$. Using this result, prove that the state

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

can be converted into any other state $|\psi\rangle^{AB}$ for $d_A = d_B = d$. Can Alice and Bob use LOCC to convert $|\Phi_d^+\rangle$ into a general mixed state?

c) For $d_A = d_B = 4$, assume that Alice and Bob share the state

$$|\psi\rangle^{AB} = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle\,.$$

As has been proven in the lecture, by using LOCC this state cannot be converted into the state

$$|\phi\rangle^{AB} = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle\,.$$

Estimate the maximal probability to convert $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ via LOCC.

Assume now that Alice and Bob can use a <u>catalyst</u>, an additional two-qubit pair in the state

$$|c\rangle^{A'B'} = \sqrt{0.6}\,|00\rangle + \sqrt{0.4}\,|11\rangle\,.$$

Prove that via LOCC Alice and Bob can convert $|\psi\rangle^{AB} \otimes |c\rangle^{A'B'}$ into $|\phi\rangle^{AB} \otimes |c\rangle^{A'B'}$.

d) Consider two states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ with $d_A = d_B = 2$. Prove that for any pair of states there exists either an LOCC conversion from $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ or from $|\phi\rangle^{AB}$ into $|\psi\rangle^{AB}$.

Problem 3 (Lecture notes Section 3)

a) For a bipartite state ρ^{AB}, the partial transpose with respect to Alice is denoted ρ^{T_A}, while ρ^{T_B} denotes partial transpose with respect to Bob. Prove that ρ^{T_A} and ρ^{T_B} have the same eigenvalues.

b) Consider two matrices M^{AB} and N^{AB} acting on the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. M^{T_B} denotes the partial transpose of M^{AB}, and N^{T_B} denotes the partial transpose of N^{AB}. Prove that

$$\mathrm{Tr}\left[M^{T_B} N\right] = \mathrm{Tr}\left[M N^{T_B}\right].$$

Hint: note that any matrix M^{AB} can be expanded as $M^{AB} = \sum_{i.j} |i\rangle\langle j| \otimes M_{ij}^B$ with some matrices M_{ij}^B acting on $\mathcal{H}_B$.

c) For $d_A = d_B = 2$ consider the state

$$\rho^{AB} = p\,|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{1}_4}{4}, \tag{1}$$

where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state and p is ranging between 0 and 1. For which values of p is the state entangled? The state (1) is also called Werner state.

d) For a bipartite state ρ^{AB} define

$$\sigma^{AB} = (U \otimes V)\rho^{AB}(U \otimes V)^\dagger$$

with local unitaries U and V. Prove that ρ^{T_A} and σ^{T_A} have the same eigenvalues.