Advanced quantum information: entanglement and nonlocality

Alexander Streltsov

3rd class
March 16, 2022

Advanced quantum information

- Every Wednesday 15:15 – 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al*., *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- 1. Homework sheet to be submitted via email by 22. March

Outline

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1. Entanglement detection
2. Applications of entanglement
   - Quantum teleportation
   - Superdense coding
3. Entanglement distillation and dilution
   - Shannon and von Neumann entropy
   - Typical sequences

Partial transposition

Partial transposition on Bob's subsystem:

$$\rho^{T_B} = \left(\sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \right)^{T_B}$$
$$= \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes \big(|k\rangle\langle l|\big)^{T}$$
$$= \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |l\rangle\langle k|$$

Partial transposition

Applying partial transposition to a separable state:

$$\rho_{\text{sep}}^{T_B} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes \big(|\phi_i\rangle\langle\phi_i|\big)^T = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i^*\rangle\langle\phi_i^*|$$

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⇒ **PPT criterion:** if ρ^{T_B} is not positive, ρ must be entangled

Partial transposition

Example. For $|\psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$ we have

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

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$$\rho^{T_A} = \begin{pmatrix} X^T & Y^T \\ \left(Y^\dagger\right)^T & Z^T \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & 0 \\ 0 & 0 & \cos\alpha\sin\alpha & 0 \\ 0 & \cos\alpha\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & \sin^2\alpha \end{pmatrix}$$

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Eigenvalues of ρ^{T_A}: $\cos^2\alpha$, $\sin^2\alpha$, $\pm|\cos\alpha\sin\alpha|$
$\Rightarrow |\psi\rangle$ is entangled for all $\alpha \neq n\frac{\pi}{2}$

Positive and completely positive maps

Positive map: linear map Λ acting on matrices such that $\Lambda(\rho)$ is positive semidefinite for any positive semidefinite matrix ρ

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For a bipartite density matrix ρ^{AB} we define

$$\mathbb{1} \otimes \Lambda(\rho^{AB}) = \mathbb{1} \otimes \Lambda\left(\sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes |k\rangle\langle l|\right) = \sum_{i,j,k,l} c_{ijkl} \, |i\rangle\langle j| \otimes \Lambda(|k\rangle\langle l|)$$

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Completely positive (CP) map: a positive map Λ such that $\mathbb{1} \otimes \Lambda(\rho^{AB})$ is positive for any positive semidefinite matrix ρ^{AB} in the extended Hilbert space of any dimension

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Not every positive map is CP (e.g. transpose)

Choi–Jamiołkowski isomorphism

Choi matrix of a linear map Λ:

$$M_\Lambda = (\mathbb{1} \otimes \Lambda) |\Phi_d^+\rangle\langle\Phi_d^+| = \frac{1}{d} \sum_{i,j} |i\rangle\langle j| \otimes \Lambda \left(|i\rangle\langle j|\right)$$

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Choi-Jamiołkowski isomorphism:

- Λ is a positive map if and only if M_Λ is an entanglement witness
- For any entanglement witness W^{AB} there exists a positive map Λ such that $W^{AB} = M_\Lambda$
- Λ is completely positive if and only if M_Λ is positive semidefinite

PPT criterion for two qubits

Proposition 3.1. For $d_A = d_B = 2$ a state ρ^{AB} is separable if and only if ρ^{T_B} is positive semidefinite.

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Proof. For any entangled state ρ^{AB} there exists an entanglement witness W^{AB} such that (see Theorem 3.1.)

$$\mathrm{Tr}\left[W^{AB}\rho^{AB}\right] < 0.$$

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$$\mathrm{Tr}\left[W^{AB}\rho^{AB}\right] < 0.$$

With the Choi-Jamiołkowski isomorphism, there also exists a positive map Λ such that

$$\mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda\,|\Phi^{+}\rangle\langle\Phi^{+}|\right)\rho^{AB}\right] < 0.$$

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Every positive qubit map can be decomposed as

$$\Lambda(\rho) = \Lambda_1^{\mathrm{CP}}(\rho) + \left[\Lambda_2^{\mathrm{CP}}(\rho)\right]^T$$

with CP maps Λ_i^{CP}.

PPT criterion for two qubits

There exists a positive map Λ such that

$$\mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda \, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] < 0.$$

Thus,

$$\begin{aligned} 0 > \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda \, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] &= \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda_1^{\mathrm{CP}} \, |\Phi^+\rangle\langle\Phi^+|\right)\rho^{AB}\right] \\ &+ \mathrm{Tr}\left[\left(\mathbb{1} \otimes \Lambda_2^{\mathrm{CP}} \, |\Phi^+\rangle\langle\Phi^+|\right)^{T_B}\rho^{AB}\right] \\ &= \mathrm{Tr}\left[X_1\rho^{AB}\right] + \mathrm{Tr}\left[X_2^{T_B}\rho^{AB}\right] \end{aligned}$$

with positive matrices $X_i = \mathbb{1} \otimes \Lambda_i^{\mathrm{CP}} \, |\Phi^+\rangle\langle\Phi^+|$.

PPT criterion for two qubits

In summary,

$$0 > \mathrm{Tr}\left[X_1 \rho^{AB}\right] + \mathrm{Tr}\left[X_2^{T_B} \rho^{AB}\right]$$

with positive matrices $X_i = \mathbb{1} \otimes \Lambda_i^{\mathrm{CP}} |\Phi^+\rangle\langle\Phi^+|$.

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In summary,

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Using

$$\mathrm{Tr}\left[X_2^{T_B}\rho^{AB}\right] = \mathrm{Tr}\left[X_2\rho^{T_B}\right]$$

we obtain

$$0 > \mathrm{Tr}\left[X_1\rho^{AB}\right] + \mathrm{Tr}\left[X_2\rho^{T_B}\right] \geq \mathrm{Tr}\left[X_2\rho^{T_B}\right]$$

Since X_2 is positive, ρ^{T_B} must have negative eigenvalues.
Q.E.D.

PPT criterion for two qubits

For larger dimensions:

PPT criterion for two qubits

For larger dimensions:

Theorem 3.2. For $d_A d_B \leq 6$ a state ρ^{AB} is separable if and only if ρ^{T_B} is positive. For all $d_A d_B > 6$ there exist entangled states which have positive partial transpose.

PPT criterion for two qubits

Exercise: For the two-qubit state

$$\rho = p\,|\Phi^+\rangle\langle\Phi^+| + (1-p)\,|\Phi^-\rangle\langle\Phi^-|$$

with $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $0 \leq p \leq 1$ determine the values of p for which the state is entangled.

PPT criterion for two qubits

Solution: Consider the density matrix

$$\rho = \frac{p}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1-p}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2p-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p-1 & 0 & 0 & 1 \end{pmatrix}$$

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$$\rho^{T_A} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2p-1 & 0 \\ 0 & 2p-1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$\rho^{T_A} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2p-1 & 0 \\ 0 & 2p-1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues of ρ^{T_A} : $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}(1-2p)$, $\frac{1}{2}(2p-1) \Rightarrow \rho$ is entangled for $p \neq \frac{1}{2}$

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CNOT gate

Controlled NOT gate (CNOT): A unitary transformation acting on two qubits (control and target) as follows

Before		**After**	
Control	Target	Control	Target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Hadamard gate

Hadamard gate is a unitary transformation on one qubit acting as follows

$$
\begin{aligned}
|0\rangle &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
|1\rangle &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}
$$

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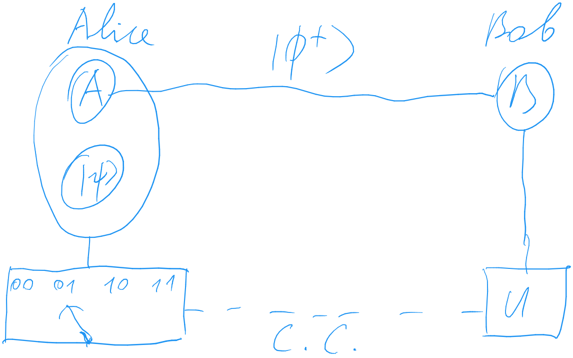
Exercise: find the matrix form of the Hadamard gate

Solution:

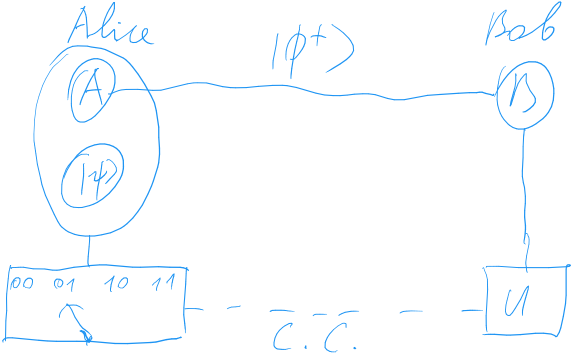
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Quantum teleportation



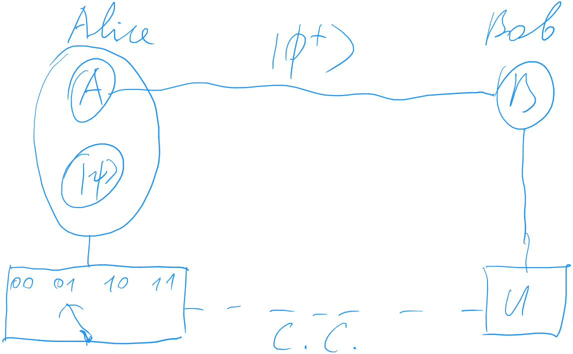
- Suppose Alice and Bob share a Bell state $|\Phi^+\rangle^{AB}$

Quantum teleportation



- Suppose Alice and Bob share a Bell state $|\Phi^+\rangle^{AB}$
- Additionally, Alice has a qubit A' in the state $|\psi\rangle^{A'} = c_0\,|0\rangle + c_1\,|1\rangle$

Quantum teleportation



- Suppose Alice and Bob share a Bell state $|\Phi^+\rangle^{AB}$
- Additionally, Alice has a qubit A' in the state $|\psi\rangle^{A'} = c_0\,|0\rangle + c_1\,|1\rangle$
- Alice can send the qubit A' to Bob by using **quantum teleportation**

Quantum teleportation

Total initial state of Alice and Bob:

$$
\begin{aligned}
|\Phi\rangle^{A'AB} &= \left(c_0\,|0\rangle^{A'} + c_1\,|1\rangle^{A'}\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle^{AB} + |11\rangle^{AB}\right) \\
&= \frac{1}{\sqrt{2}}\left[c_0\,|0\rangle\left(|00\rangle + |11\rangle\right) + c_1\,|1\rangle\left(|00\rangle + |11\rangle\right)\right]
\end{aligned}
$$

Quantum teleportation

Total initial state of Alice and Bob:

$$\begin{aligned}|\Phi\rangle^{A'AB} &= \left(c_0 |0\rangle^{A'} + c_1 |1\rangle^{A'}\right) \otimes \frac{1}{\sqrt{2}} \left(|00\rangle^{AB} + |11\rangle^{AB}\right) \\ &= \frac{1}{\sqrt{2}} \left[c_0 |0\rangle \left(|00\rangle + |11\rangle\right) + c_1 |1\rangle \left(|00\rangle + |11\rangle\right)\right]\end{aligned}$$

Alice performs a CNOT gate on her qubits $A'A$:

$$|\Phi'\rangle = \frac{1}{\sqrt{2}} \left[c_0 |0\rangle \left(|00\rangle + |11\rangle\right) + c_1 |1\rangle \left(|10\rangle + |01\rangle\right)\right]$$

Quantum teleportation

$$|\Phi'\rangle = \frac{1}{\sqrt{2}} \left[c_0 |0\rangle \left(|00\rangle + |11\rangle \right) + c_1 |1\rangle \left(|10\rangle + |01\rangle \right) \right]$$

Alice applies a Hadamard gate to A':

$$\begin{aligned} |\Phi''\rangle &= \frac{1}{2} \left[c_0 \left(|0\rangle + |1\rangle \right) \left(|00\rangle + |11\rangle \right) + c_1 \left(|0\rangle - |1\rangle \right) \left(|10\rangle + |01\rangle \right) \right] \\ &= \frac{1}{2} \left[|00\rangle \left(c_0 |0\rangle + c_1 |1\rangle \right) + |01\rangle \left(c_0 |1\rangle + c_1 |0\rangle \right) \right. \\ &\left. + |10\rangle \left(c_0 |0\rangle - c_1 |1\rangle \right) + |11\rangle \left(c_0 |1\rangle - c_1 |0\rangle \right) \right] \end{aligned}$$

Quantum teleportation

$$|\Phi''\rangle = \frac{1}{2}\left[|00\rangle\left(c_0|0\rangle + c_1|1\rangle\right) + |01\rangle\left(c_0|1\rangle + c_1|0\rangle\right) \right.$$
$$\left. + |10\rangle\left(c_0|0\rangle - c_1|1\rangle\right) + |11\rangle\left(c_0|1\rangle - c_1|0\rangle\right)\right]$$

Alice measures A' and A in the computational basis $\{|0\rangle, |1\rangle\}$:

Alice's outcome	State of B
00	$c_0|0\rangle + c_1|1\rangle$
01	$c_0|1\rangle + c_1|0\rangle$
10	$c_0|0\rangle - c_1|1\rangle$
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Quantum teleportation

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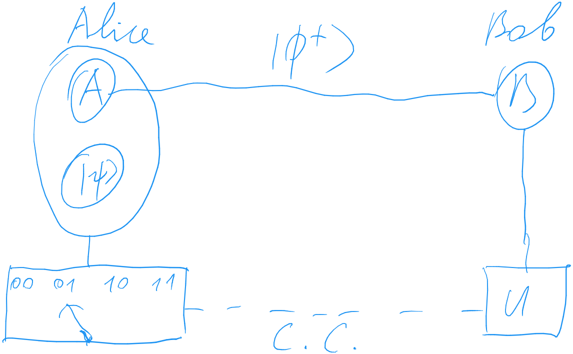
Bob performs a correction on his qubit depending on Alice's measurement:

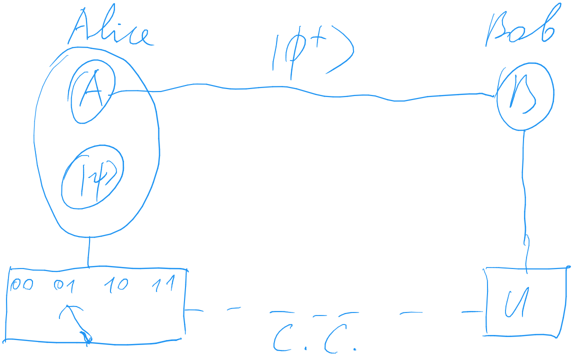
Alice's outcome	State of B	Correction	State of B after correction
00	$c_0 \|0\rangle + c_1 \|1\rangle$	$\mathbb{1}$	$c_0 \|0\rangle + c_1 \|1\rangle$
01	$c_0 \|1\rangle + c_1 \|0\rangle$	σ_x	$c_0 \|0\rangle + c_1 \|1\rangle$
10	$c_0 \|0\rangle - c_1 \|1\rangle$	σ_z	$c_0 \|0\rangle + c_1 \|1\rangle$
11	$c_0 \|1\rangle - c_1 \|0\rangle$	$i\sigma_y$	$c_0 \|0\rangle + c_1 \|1\rangle$

Quantum teleportation



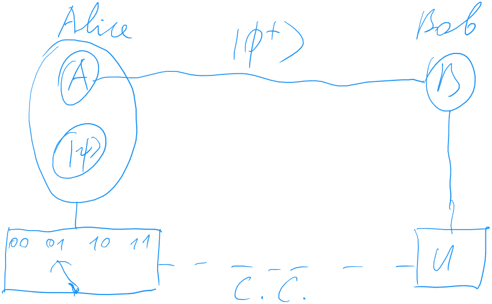
- Protocol does not depend on the state to be teleported

Quantum teleportation



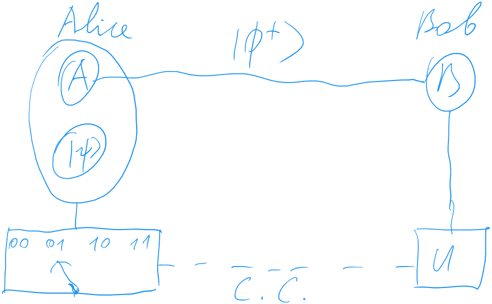
- Protocol does not depend on the state to be teleported
- Bell state $|\Phi^+\rangle^{AB}$ is destroyed in this procedure, thus teleportation of one qubit consumes one Bell state

Quantum teleportation



For $d > 2$:

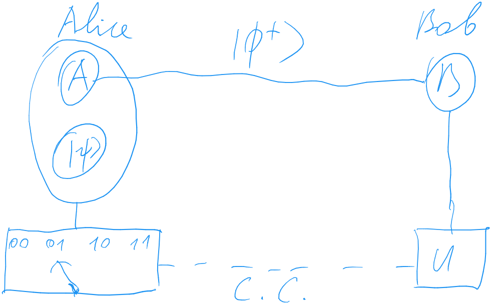
Quantum teleportation



For $d > 2$:

- if $d = 2^n$ for some $n \in \mathbb{N}$: A' can be treated as n-qubit system: $A' = A'_1 A'_2 \dots A'_n$

Quantum teleportation



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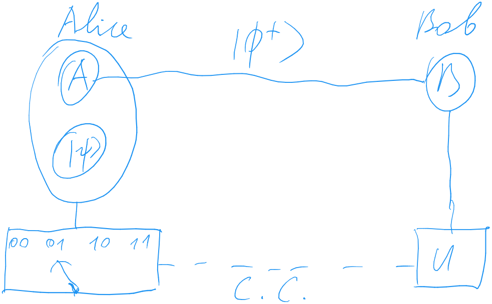
Quantum teleportation



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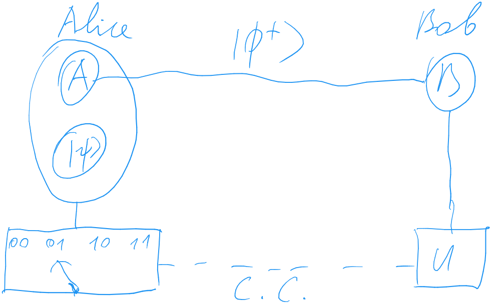
- Consumes $n = \log_2 d$ Bell states

Quantum teleportation



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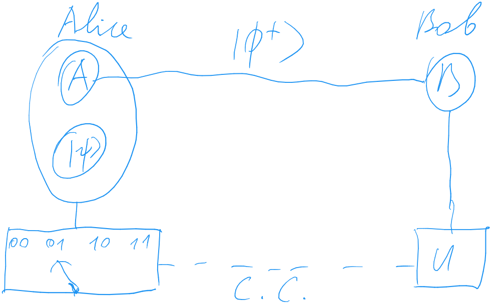

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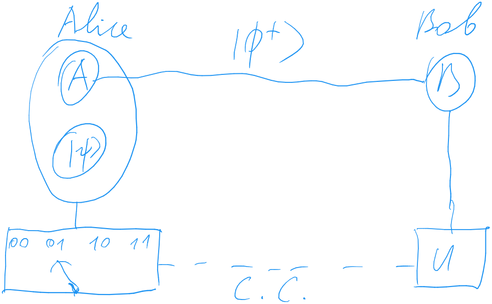
Quantum teleportation



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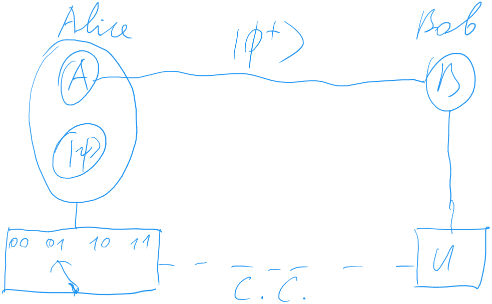
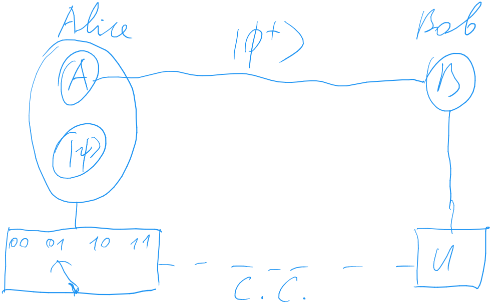
Quantum teleportation



For $d > 2$:

- if $d \neq 2^n$ for any $n \in \mathbb{N}$, define $n = \lceil \log_2 d \rceil$
- $|\psi\rangle^{A'} = \sum_{i=0}^{2^n-1} c_i \, |i\rangle^{A'}$ with $c_i = 0$ for $i \geq d$
- A' can be treated as n-qubit system: $A' = A'_1 A'_2 \ldots A'_n$
- Teleportation of A' by teleporting each of the qubits A'_i

Quantum teleportation



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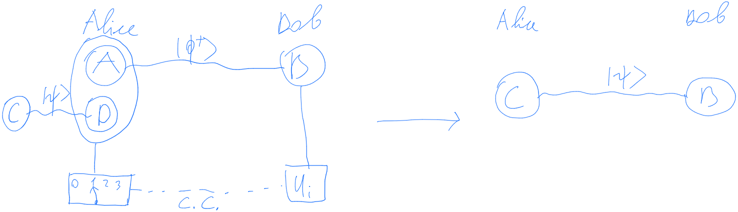
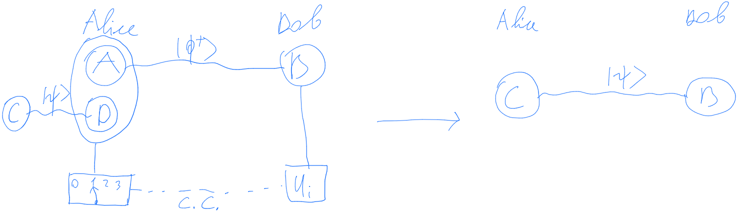
- Consumes $n = \lceil \log_2 d \rceil$ Bell states

Quantum teleportation



- Quantum teleportation can also be applied to teleport a part of Alice's subsystem

Quantum teleportation



- Quantum teleportation can also be applied to teleport a part of Alice's subsystem
- **Proposition 4.1.** For a state

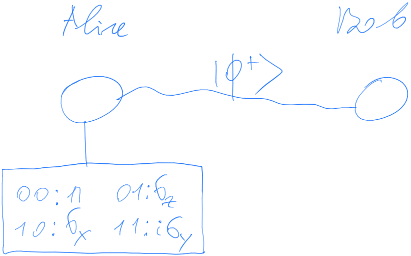
$$|\psi\rangle^{CD} = \sum_{i=0}^{k-1} \sqrt{\lambda_i}\, |i\rangle^C \otimes |i\rangle^D$$

with k nonzero Schmidt coefficients the teleportation of D can be done by consuming $\lceil \log_2 k \rceil$ Bell states.

Outline

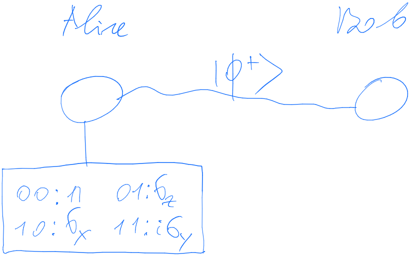
1. Entanglement detection
2. Applications of entanglement
   - Quantum teleportation
   - Superdense coding
3. Entanglement distillation and dilution
   - Shannon and von Neumann entropy
   - Typical sequences

Superdense coding



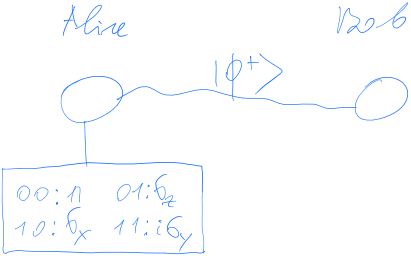
- Suppose that Alice and Bob share two qubits in the state $|\Phi^+\rangle$
- They can use $|\Phi^+\rangle$ to communicate two bits of information with a single qubit via the following procedure

Superdense coding



1. Alice applies a unitary on her qubit, depending on which two bits she wants to send to Bob

Superdense coding



Resulting states are

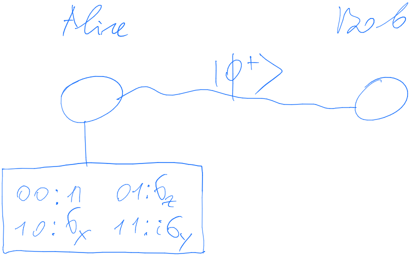
$$00 : |\Phi^+\rangle \rightarrow (\mathbb{1} \otimes \mathbb{1})\,|\Phi^+\rangle = |\Phi^+\rangle\,,$$

$$01 : |\Phi^+\rangle \rightarrow (\sigma_z \otimes \mathbb{1})\,\frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle) = |\Phi^-\rangle\,,$$

$$10 : |\Phi^+\rangle \rightarrow (\sigma_x \otimes \mathbb{1})\,\frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\,(|10\rangle + |01\rangle) = |\Psi^+\rangle\,,$$

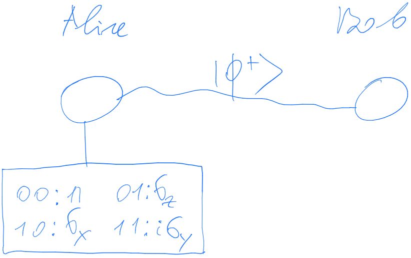
$$11 : |\Phi^+\rangle \rightarrow \left(i\sigma_y \otimes \mathbb{1}\right)\,\frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\,(-|10\rangle + |01\rangle) = |\Psi^-\rangle\,.$$

Superdense coding



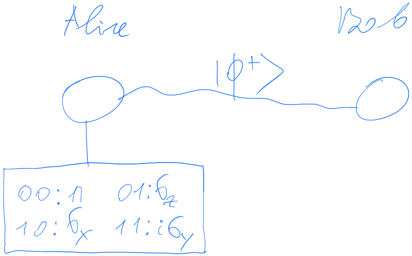
2. Alice sends her qubit to Bob, who is now in possession of one of the four Bell states

Superdense coding



3. Bob applies a von Neumann measurement in the maximally entangled basis. From his outcome, he can directly read off the two bits encoded by Alice.

Superdense coding



Two bits is the maximal amount of classical information that one qubit can carry

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Shannon and von Neumann entropy

Consider an integer random variable x with probability distribution $p(x)$. A sequence of **independent and identically distributed** variables x_i has probability distribution

$$p(x_1, \ldots, x_m) = p(x_1)p(x_2)\ldots p(x_m).$$

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$$H(p(x)) = -\sum_x p(x)\log_2 p(x)$$

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Von Neumann entropy of a quantum state ρ with eigenvalues λ_i:

$$S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i$$

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Typical sequences

$P(\text{heads}) = 2/3$

$P(\text{tails}) = 1/3$

- Consider a sequence $x_1, x_2, \ldots, x_m$ coming from m flips of a biased coin ($0 = \text{heads}$, $1 = \text{tails}$)

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ϵ-typical sequence: sequence of independent and identically distributed random variables x_i such that

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Exercise: For $\epsilon = 0.01$ and $m = 10$ is the sequence $1, 1, \ldots, 1, 1$ ϵ-typical?

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- For $m = 10$ we have $p(1, 1, \ldots, 1, 1) = \frac{1}{3^{10}} \approx 2 \times 10^{-5}$

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- $\Rightarrow 1, 1, \ldots, 1, 1$ is not ϵ-typical