Advanced quantum information: entanglement and nonlocality

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2nd class
March 9, 2022

Advanced quantum information

- Every Wednesday 15:15 - 17:00
- Literature:
  - Nielsen and Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2012)
  - Horodecki *et al*., *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)
- Howework and lecture notes:
  http://qot.cent.uw.edu.pl/teaching/
- Homework to be submitted via email as a single pdf

Schmidt decomposition

- For any pure state $|\psi\rangle^{AB}$ there exists a product basis $\{|i\rangle \otimes |j\rangle\}$ such that
$$|\psi\rangle^{AB} = \sum_i \sqrt{\lambda_i}\, |i\rangle \otimes |i\rangle$$
with $\lambda_i \geq 0$

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- Schmidt coefficients are equal to the **eigenvalues of the reduced states** $\mathrm{Tr}_A[|\psi\rangle\langle\psi|^{AB}]$ and $\mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$

Outline

1. Theory of quantum entanglement
   - Local operations and classical communication
   - Pure state conversion via LOCC
   - Probabilistic conversion and catalysis
   - Bell states
   - Entanglement for mixed states

2. Entanglement detection
   - Entanglement witnesses

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Local operations and classical communication (LOCC)

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4. The outcome j of Bob's measurement is communicated classically to Alice.
5. Alice performs a local measurement on her subsystem which can depend on all outcomes of all previous measurements, and the process starts over at step 2.

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Pure state conversion via LOCC

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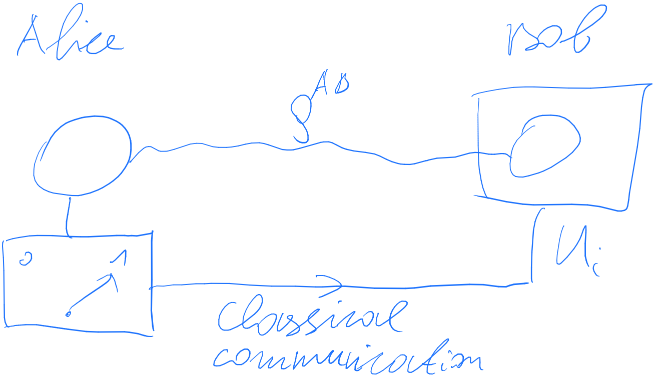
Proposition 2.1. Suppose $|\psi\rangle^{AB}$ can be transformed into $|\phi\rangle^{AB}$ via LOCC. Then this transformation can be achieved by a protocol involving just the following steps: Alice performs a measurement with Kraus operators $\{K_j\}$, sends the result j to Bob, who applies a conditional unitary U_j on his system.

Pure state conversion via LOCC



Proof of Proposition 2.1

Let $K_j = \sum_{k,l} K_{j,kl} |k\rangle\langle l|$ be a Kraus operator of Bob expanded in the Schmidt basis of $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$. The post-measurement state $|\mu_j\rangle$ is given as

$$|\mu_j\rangle = \frac{\mathbb{1} \otimes K_j |\psi\rangle}{\sqrt{p_j}} = \frac{\sum_{k,l} K_{j,kl} \sqrt{\lambda_l} |l\rangle \otimes |k\rangle}{\sqrt{p_j}}$$

with probability

$$p_j = \langle\psi| \mathbb{1} \otimes K_j^\dagger K_j |\psi\rangle = \sum_{k,l} \lambda_l |K_{j,kl}|^2.$$

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with probability

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Assume now that instead Alice performs a measurement with Kraus operator $L_j = \sum_{k,l} K_{j,kl} |k\rangle\langle l|$, leading to the state

$$|\nu_j\rangle = \frac{L_j \otimes \mathbb{1} |\psi\rangle}{\sqrt{p_j}} = \frac{\sum_{k,l} K_{j,kl} \sqrt{\lambda_l} |k\rangle \otimes |l\rangle}{\sqrt{p_j}}$$

with the same probability p_j.

Proof of Proposition 2.1

Note that $|\mu_j\rangle$ and $|\nu_j\rangle$ are the same up to interchanging A and B, which by Schmidt decomposition implies that

$$|\mu_j\rangle = \sum_i \sqrt{\alpha_{ij}} \left(U_j |i\rangle \right) \otimes \left(V_j |i\rangle \right),$$

$$|\nu_j\rangle = \sum_i \sqrt{\alpha_{ij}} \left(V_j |i\rangle \right) \otimes \left(U_j |i\rangle \right)$$

for some $\alpha_{ij} \geq 0$ and local unitaries U_j and V_j, and thus

$$|\mu_j\rangle = \left(U_j V_j^\dagger \otimes V_j U_j^\dagger \right) |\nu_j\rangle .$$

Thus, Bob performing a measurement $\{K_j\}$ on $|\psi\rangle$ is equivalent to Alice performing a measurement $\{U_j V_j^\dagger L_j\}$, followed by Bob performing the unitary $V_j U_j^\dagger$.

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Proof of Proposition 2.1

- A measurement by Bob on a pure state can be simulated by a measurement by Alice, and a conditional unitary by Bob
- If Alice and Bob perform an LOCC protocol consisting of many rounds of measurements and classical communication, we replace each round involving Bob's measurement by a corresponding measurement on Alice's side
- In this way, any LOCC protocol transforming $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ can be simulated by a single measurement of Alice, followed by conditional unitary on Bob's side

Pure state conversion via LOCC

Majorization:

- Consider two real d-dimensional vectors $\vec{x}$ and $\vec{y}$ with elements in decreasing order

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- Consider two real d-dimensional vectors $\vec{x}$ and $\vec{y}$ with elements in decreasing order
- Then $\vec{x} \prec \vec{y}$ if
$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$
for all $k \in [1, d-1]$, and $\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$

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- For a Hermitian matrix H let $\vec{\lambda}_H$ be the vector of eigenvalues of H in decreasing order
- For two Hermitian matrices H and K we write $H \prec K$ if $\vec{\lambda}_H \prec \vec{\lambda}_K$

Pure state conversion via LOCC

Proposition 2.2. Let H and K be Hermitian matrices. Then $H \prec K$ if and only if there is a probability distribution p_j and unitary matrices U_j such that

$$H = \sum_j p_j U_j K U_j^\dagger.$$

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Theorem 2.1. (Nielsen's Theorem) There exists an LOCC protocol transforming $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ if and only if $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$, where $\vec{\lambda}_\psi$ denotes the vector with eigenvalues of the reduced state $\mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$ in decreasing order.

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- Suppose $|\psi\rangle^{AB}$ can be transformed into $|\phi\rangle^{AB}$ via LOCC.
- By proposition 2.1, the transformation is achieved if Alice applies a measurement with local Kraus operators $\{K_j\}$ and Bob applies local unitaries $\{U_j\}$.
- After Alice's measurement, the total post-measurement state is equal to $|\phi\rangle^{AB}$ up to local unitaries on Bob's side:

$$K_j \otimes \mathbb{1} \, |\psi\rangle^{AB} = \sqrt{p_j} \mathbb{1} \otimes U_j^\dagger \, |\phi\rangle^{AB} .$$

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- Defining $\rho_\psi = \mathrm{Tr}_B[|\psi\rangle\langle\psi|^{AB}]$ and $\rho_\phi = \mathrm{Tr}_B[|\phi\rangle\langle\phi|^{AB}]$, we get

$$K_j \rho_\psi K_j^\dagger = p_j \rho_\phi$$

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- By proposition 2.2 we have $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$.

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for some probabilities p_j and unitaries U_j.

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- It holds that

$$\sum_j K_j^\dagger K_j = \rho_\psi^{-1/2} \left(\sum_j p_j U_j \rho_\phi U_j^\dagger \right) \rho_\psi^{-1/2} = \rho_\psi^{-1/2} \rho_\psi \rho_\psi^{-1/2} = \mathbb{1},$$

  thus K_j are valid Kraus operators.

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- When Alice applies the measurement $\{K_j\}$ to the total state $|\psi\rangle^{AB}$, she obtains the reduced state ρ_ϕ with probability p_j.
- Since all purifications of ρ_ϕ are equivalent up to unitary on Bob's side, it follows that there exist unitaries U_j on Bob's side such that
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$$K_j \otimes \mathbb{1} \, |\psi\rangle^{AB} = \sqrt{p_j} \mathbb{1} \otimes U_j \, |\phi\rangle^{AB} \, .$$

- Thus, if Alice applies measurement $\{K_j\}$ to the state $|\psi\rangle^{AB}$, communicates the measurement outcome j to Bob, and he performs $U_j^\dagger$, they achieve the conversion $|\psi\rangle^{AB} \to |\phi\rangle^{AB}$.

Pure state conversion

Exercise: Consider the states

$$|\psi\rangle^{AB} = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle$$
$$|\phi\rangle^{AB} = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle$$

Is the conversion $|\psi\rangle^{AB} \to |\phi\rangle^{AB}$ or $|\phi\rangle^{AB} \to |\psi\rangle^{AB}$ possible via LOCC?

Hint: Check if $\vec{\lambda}_\psi \prec \vec{\lambda}_\phi$, recalling that $\vec{x} \prec \vec{y}$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

for all $k \in [1, d-1]$, and $\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$

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- Definition for general ρ^{AB} and σ^{AB}

$$P(\rho^{AB} \to \sigma^{AB}) = \max_{\{K_i\}} \left\{ \mathrm{Tr}\left[\sum_i K_i \rho^{AB} K_i^\dagger\right] : \sigma^{AB} = \frac{\sum_i K_i \rho^{AB} K_i^\dagger}{\mathrm{Tr}\left[\sum_i K_i \rho^{AB} K_i^\dagger\right]} \right\}$$

- Maximum is taken over all (incomplete) sets of Kraus operators $\{K_i\}$ which are implementable via LOCC

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- Pure states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$:

$$P\left(|\psi\rangle^{AB} \to |\phi\rangle^{AB}\right) = \min_{l \in [1,n]} \frac{\sum_{i=l}^{n} \alpha_i}{\sum_{j=l}^{n} \beta_j}$$

- α_i and β_j are the Schmidt coefficients of $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ sorted in decreasing order

Catalytic conversion

If there is no LOCC protocol such that

$$|\psi\rangle^{AB} \to |\phi\rangle^{AB},$$

there might be a catalyst state $|c\rangle^{A'B'}$ such that

$$|\psi\rangle^{AB} \otimes |c\rangle^{A'B'} \to |\phi\rangle^{AB} \otimes |c\rangle^{A'B'}$$

Outline

Bell states

Bell states (or EPR states):

$$
\begin{aligned}
|\Phi^+\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\\
|\Phi^-\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
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- Reduced state of any Bell state: $\frac{1}{2}\mathbb{1}_2$
- For any single-qubit state ρ it holds $\frac{1}{2}\mathbb{1}_2 \prec \rho$
- Theorem 2.1. $\Rightarrow$ **any Bell state can be converted into any two-qubit pure state via LOCC**

Maximally entangled states

Bell states are also called **maximally entangled states**

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- For $d_A = d_B = d$ a state $|\Psi_d\rangle$ is maximally entangled if and only if

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- All maximally entangled states are equivalent to

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$$

up to local unitary on one side:

$$|\Psi_d\rangle = \big(U\otimes\mathbb{1}\big)\,|\Phi_d^+\rangle = \big(\mathbb{1}\otimes V\big)\,|\Phi_d^+\rangle$$

Outline

Entanglement for mixed states

Separable mixed states:

$$\rho_{\mathrm{sep}}^{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

with $p_i \geq 0$, $\sum_i p_i = 1$, $|\psi_i\rangle \in \mathcal{H}_A$ and $|\phi_i\rangle \in \mathcal{H}_B$.

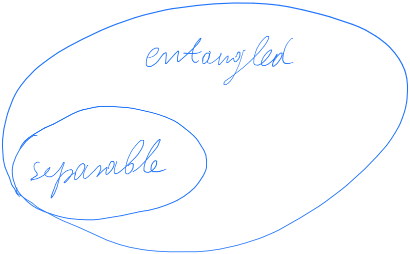
Entanglement for mixed states

Separable mixed states:

$$\rho_{\mathrm{sep}}^{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

with $p_i \geq 0$, $\sum_i p_i = 1$, $|\psi_i\rangle \in \mathcal{H}_A$ and $|\phi_i\rangle \in \mathcal{H}_B$.

States which are not separable are called **entangled**



Outline

1. Theory of quantum entanglement
   - Local operations and classical communication
   - Pure state conversion via LOCC
   - Probabilistic conversion and catalysis
   - Bell states
   - Entanglement for mixed states

2. Entanglement detection
   - Entanglement witnesses

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Entanglement witnesses

Entanglement witness: Hermitian matrix W^{AB} such that

$$\mathrm{Tr}\left[W^{AB}\left(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|\right)\right] = \left(\langle\psi| \otimes \langle\phi|\right) W^{AB} \left(|\psi\rangle \otimes |\phi\rangle\right) \geq 0$$

for any $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$

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- For any separable state ρ_{sep}^{AB} we have

$$\begin{aligned}\mathrm{Tr}\left[W^{AB}\rho_{\mathrm{sep}}^{AB}\right] &= \mathrm{Tr}\left[W^{AB}\left(\sum_i p_i\,|\psi_i\rangle\langle\psi_i|\otimes|\phi_i\rangle\langle\phi_i|\right)\right] \\ &= \sum_i p_i\,\mathrm{Tr}\left[W^{AB}\left(|\psi_i\rangle\langle\psi_i|\otimes|\phi_i\rangle\langle\phi_i|\right)\right] \geq 0\end{aligned}$$

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- If $\operatorname{Tr}\left[W^{AB}\rho^{AB}\right] < 0$, the state ρ^{AB} must be entangled

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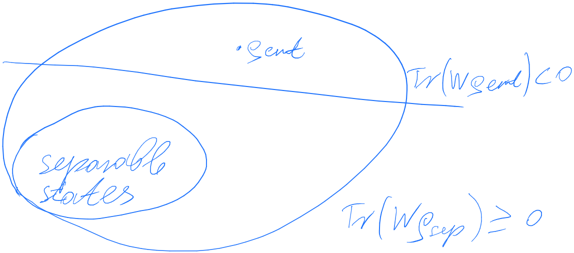

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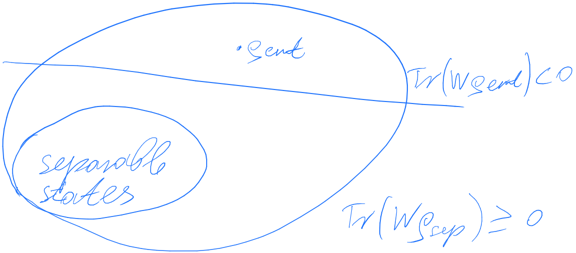

Interpretation of W^{AB}: observable with expectation value $\mathrm{Tr}\left[W^{AB}\rho^{AB}\right]$

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Example. Swap operation for $d_A = d_B$:

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- $\Rightarrow W^{AB}$ detects entanglement in $|\Psi^-\rangle$